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THE ROLE OF THE HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION: REFLECTIONS AND EXAMPLES

Giorgio T. Bagni
Nucleo di Ricerca in Didattica della Matematica, Department of Mathematics, University of Bologna (Italy)
gtbagni@tin.it

Abstract: *The effectiveness of the use of history of mathematics in mathematics education is worthy of careful research. In this paper the introduction of the group concept to an experimental sample of students aged 16-18 years by an historical example drawn from Bombelli's Algebra (1572) is described. A second sample of students was given a parallel introduction through a Cayley table. Both groups were asked the same test questions and their responses examined and compared.*

1. Bombelli's *Algebra* (1572) and imaginary numbers

Several authors have shown that the history of mathematics can be drawn on by teachers in the presentation of many mathematical topics to the benefit of pupils. It follows that research into the role of history of mathematics in teaching is legitimately considered a part of research into mathematics education (many references can be mentioned; for example: Jahnke, 1991, 1995 and 1996).

Of course we must consider the educational use of history of mathematics at different levels, and these levels can lead to different educational interventions. For example, according to the conception of the mathematics education as *thought transference*, the main purpose of the educational research is *improvement of teaching*. The presentation of mathematical topics using historical references is consistent with this approach. Of course, the effectiveness of the historical introduction will be judged with respect to pupils' learning.

In this paper, we consider an important topic of the curriculum of both high school (for students aged 16-18 years) and undergraduate mathematics, namely the group concept.

Rafael Bombelli of Bologna (1526-1572) was the author of *Algebra*, published twice, in 1572 and in 1579 (the dispute between G. Cardan and N. Tartaglia about the resolution of cubic equations is well known; Scipio Del Ferro was remembered in Bombelli's *Algebra*: manuscript B.1569, Archiginnasio, Bologna).

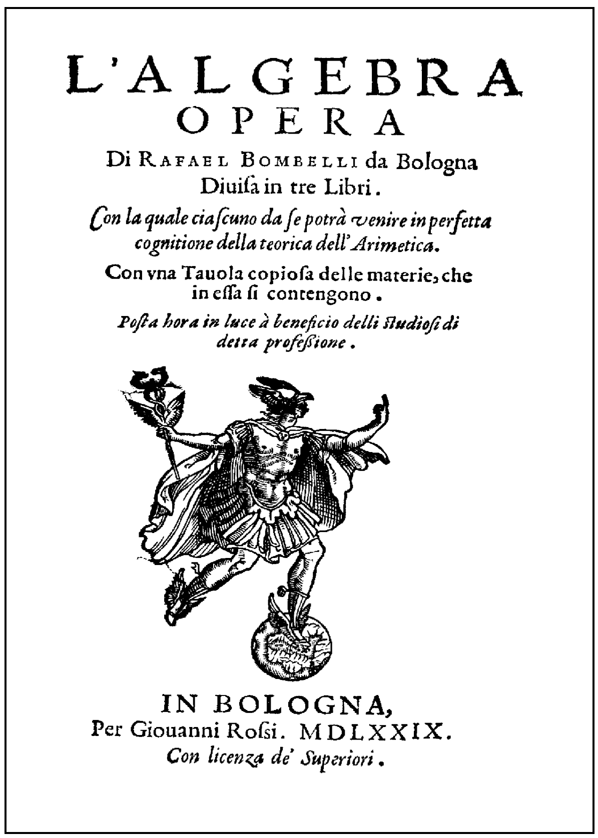
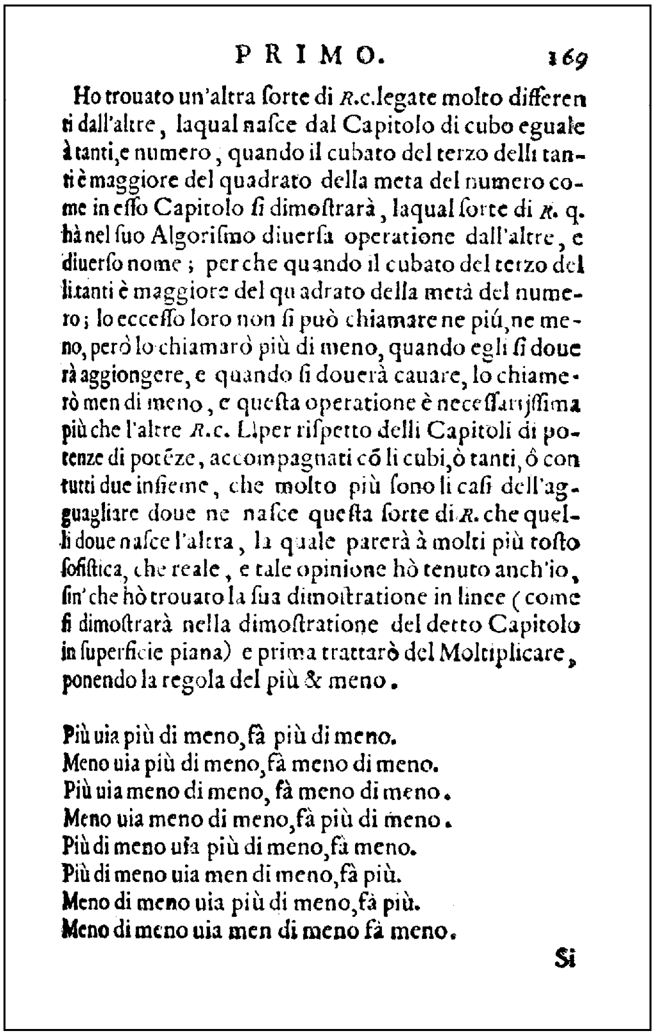

L'ALGEBRA
OPERA
Di RAFAEL BOMBELLI da Bologna
Diuiſa in tre Libri.
Con la quale ciaſcuno da ſe potrà venire in perfetta cognitione della teorica dell'Arimetica.
Con vna Tauola copioſa delle materie, che in eſſa ſi contengono.
Poſta hora in luce à beneficio delli ſtudioſi di detta profeſſione.
IN BOLOGNA,
Per Giouanni Roſsi. MDLXXIX.
Con licenza de' Superiori.

Fig. 1 Bombelli's *Algebra* (1572-1579)

In *Algebra*'s 1st Book, Bombelli introduced the terms *più di meno* (*pdm*) and *meno di meno* (*mdm*) to represent +*i* and –*i* and gave some "basic rules". Let us consider them in Bombelli's original words (p. 169):

"Più via più di meno, fa più di meno. Meno via più di meno, fa meno di meno.
Più via meno di meno, fa meno di meno. Meno via meno di meno, fa più di meno.
Più di meno via più di meno, fa meno. Più di meno via men di meno, fa più.
Meno di meno via più di meno, fa più. Meno di meno via men di meno, fa meno"



PRIMO. 169

Ho trouato un'altra ſorte di R.c.legate molto differenti dall'altre, laqual naſce dal Capitolo di cubo eguale à tanti,e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della meta del numero come in eſſo Capitolo ſi dimoſtrarà, laqual ſorte di R. q. hà nel ſuo Algoriſmo diuerſa operatione dall'altre, e diuerſo nome; perche quando il cubato del terzo delli tanti è maggiore del quadrato della metà del numero; lo eceſſo loro non ſi può chiamare ne piú,ne meno, però lo chiamarò più di meno, quando egli ſi douerà aggiongere, e quando ſi douerà cauare, lo chiamerò men di meno, e queſta operatione è neceſſarijſſima più che l'altre R.c. L per riſpetto delli Capitoli di potenze di poteze, accompagnati cō li cubi,ò tanti, ô con tutti due inſieme, che molto più ſono li caſi dell'agguagliare doue ne naſce queſta ſorte di R. che quelli doue naſce l'altra, la quale parerà à molti più toſto ſofiſtica, che reale, e tale opinione hò tenuto anch'io, ſin'che hò trouato la ſua dimoſtratione in linee (come ſi dimoſtrarà nella dimoſtratione del detto Capitolo in ſuperficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno, fà più di meno.
Più di meno uia più di meno, fà meno.
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.

Si

Fig. 2 From Bombelli's *Algebra*

Now, let us translate:

"Più" → +1; "Meno" → –1; "Più di meno" → $+i$; "Meno di meno" → $-i$; "Via" → · (multiplication); "Fa" → =.

So we can write:

$(+1)\cdot\ (+i) = +i$ $(-1)\cdot\ (+i) = -i$
$(+1)\cdot\ (-i) = -i$ $(-1)\cdot\ (-i) = +i$
$(+i)\cdot\ (+i) = -1$ $(+i)\cdot\ (-i) = +1$
$(-i)\cdot\ (+i) = +1$ $(-i)\cdot\ (-i) = -1$

First of all, we underline the importance of the linguistic aspect: for example, we have translated the term "Fa" with the symbol "="; however the modern equality symbol can be referred to a relation "in two directions", while the term "Fa" means that the result of the multiplication in the first member is written in the second member. This can be confirmed by further research.

Moreover, in *Algebra* we find (p. 70):

"Più via più fa più. Meno via meno fa più.
Più via meno fa meno. Meno via più fa meno".

We can express these in the following way:

$(+1)\cdot\ (+1) = +1$ $(-1)\cdot\ (+1) = -1$
$(+1)\cdot\ (-1) = -1$ $(-1)\cdot\ (-1) = +1$

So we can write the following *Cayley table*:

x	+1	-1	+*i*	-*i*
+1	+1	-1	+*i*	-*i*
-1	-1	+1	-*i*	+*i*
+*i*	+*i*	-*i*	-1	+1
-*i*	-*i*	+*i*	+1	-1

It can be interpreted as the multiplicative group $(\{+1; -1; +i; -i\}; \cdot\ )$ of the fourth roots of unity, a finite Abelian group (it is well known that it is a cyclic group and it can be generated either by i either by $-i$; Bombelli did not notice explicitly this property; moreover he did not notice this for its cyclic subgroup generated by –1).

Of course, in Bombelli's *Algebra* we cannot find a modern introduction either of complex numbers or of the formal notion of a group: Bombelli just indicated some mathematical objects in order to solve cubic equations. These ideas were not immediately accepted following the publication of Cardan's and Bombelli's works. Bombelli himself was initially doubtful and wrote in *Algebra*, p. 169: "I found another kind of cubic root... and I did not consider it real, until I have found its proof"; let us underline that Bombelli considered fundamental the geometric proof of algebraic statements. However the *formal* introduction of *i* in Bombelli's *Algebra* is important and modern (Bourbaki, 1960, pp. 91-92). We notice, according to A. Sfard, that complex numbers were introduced simply as an *operational* concept:

"*Cardan's prescriptions for solving equations of the third and fourth order, published in 1545, involved [...] even finding roots of what is today called negative numbers. Despite the widespread use of these algorithms, however, mathematicians refused to accept their by-products [...] The symbol [square root of -1 was] initially considered nothing more than an abbreviation for certain 'meaningless' numerical operations. It came to designate a fully fledged mathematical object only after mathematicians got accustomed to these strange but useful kinds of computation*" (Sfard, 1991, p. 12).

We can consider the idea of group in a similar way: surely it would be incorrect to ascribe to Bombelli an explicit awareness of the group concept, three centuries before Galois and Dedekind. However, we can state that he (implicitly) introduced – in action – one of the most important concepts of mathematics.

2. Educational problems: the focus of our work

In an important paper, E. Dubinsky, J. Dautermann, U. Leron and R. Zazkis opened "a discussion concerning the nature of the knowledge about abstract algebra, in particular group theory, and how an individual may develop an understanding of various topics in this domain" (Dubinsky, & al., 1994, p. 267).

Let us now consider only the group concept (in the article quoted we can find interesting considerations of many algebraic notions: the concepts leading up to "quotient group", for instance, are: "group", "subgroup", "coset", "coset product" and "normality": Dubinsky & al., 1994, p. 292). The authors write: "An individual's knowledge of the concept of group should include an understanding of various mathematical properties and constructions independent of particular examples, indeed including groups consisting of undefined elements and a binary operation satisfying the axioms" (Dubinsky & Al., 1994, p. 268; Leron & Dubinsky, 1995; as regards the teaching procedure using the computer software ISETL, presented in Dubinsky & Al., 1994, see for example: Dubinsky & Leron, 1994).

In a recent paper (1996), B. Burn strongly emphasises that the notion of group in Dubinsky & al. (1994) is introduced by formal definitions. In particular, a group is "a set with a binary operation satisfyi ng four axioms [...] They espouse a set-theoretic viewpoint" (Burn, 1996, p. 375). Then Burn notes that "a set -theoretic analysis is a twentieth-century analysis performed upon the mathematics of earlier centuries as well as our own" (Burn, 1996, p. 375); so he suggests a pre-axiomatic start to group theory (Burn, 1996, p. 375; for example, the author quotes: Jordan & Jordan, 1994). So, according to Burn, the group concept can be introduced *before* offering axioms (he points out the importance of geometric symmetries: Burn, 1996, p. 377; Burn, 1985; as regards fundamental concepts of group theory he quotes: Freudenthal, 1973).

In another recent paper (1997), E. Dubinsky & al. assert that "seeing the general in particular is one of the most mysterious and difficult learning tasks students have to perform" (Dubinsky & al., 1997, p. 252; Mason & Pimm, 1984); the authors mention students' difficulties with permutations and symmetries (Asiala & Al., 1996; they quote: Breidenbach & Al., 1991; Zazkis & Dubinsky, 1996; as regards visualisation: Zazkis & Al., 1996). So a very important question is the following: is it possible (and useful) to introduce the group concept by a pre-axiomatic first treatment?

In this paper we do not claim to give a full answer to this question: rather we attempt to contribute to knowledge of the development of students' understanding of the group concept, with reference to historical examples. For instance, can the consideration of the group mentioned above help students in the comprehension of the group concept? In particular, will consideration of Bombelli's "basic rules" bring *all* the properties that are fundamental to the group concept to pupils' awareness?

As indicated above, we operate on teaching to improve its quality by *thought transference*; but some reactions, especially those in pupils' minds, are inferred, they are plausible rather than certain. We proposed an example in the historical sphere, in order that students will "learn" in this sphere, but so that the knowledge achi eved will *not* be confined to the historical sphere: evolution to different spheres is necessary. A problem that can limit the efficacy of mathematics education as (merely) *thought transference* is as follows: if we operate (only) on teaching, are we *sure* that a correct evolution will take place in the students?

In what follows we shall examine students' behaviour in response to a comparitive teaching experiment carried out with two samples of high school students. To the first sample, we simply quoted Bombelli's "basic rules"; in the second one we gave the *Cayley table*. We wanted to find out if the four properties used in the definition of group (according to Burn, they are introduced by the terms: "closed", "associative", "identity" and "inverse": Burn, 1 996, p. 372) are acquired by students from these introductions.

3. The group concept from history to mathematics education

The sample comprised the students of three classes from the 3rd *Liceo Scientifico* (pupils aged 16-17 years), 68 pupils, and of three classes from the 4th *Liceo scientifico* (pupils aged 17-18 years), 71 pupils (total: 139 pupils), in Treviso (Italy). At the time of the experiment, pupils knew the definition of i ($i^2 = -1$); they did *not* know the group concept.

We divided (at random) every class into two parts, referred to as A and B; then we gave the following cards to the students. In the card given to the students in part A (total 68 students) we just quoted Bombelli's "basic rules"; in the card given to the students of the part B (total 71 students), we did not quote Bombelli's "basic rules" but we gave *Cayley table*. (See Cards A and B illustrated below.) By the test questions, we wanted to find out if consideration of a simple historical example (without an axiomatic set-theoretic viewpoint) is useful in introducing the group concept; moreover, we wanted to find out the difference between the results obtained by students who were given a group description by Bombelli's 'basic rules" and the results obtained by students who had received the *Cayley table*.

As regards the four properties used in the definition of a group, we expected that closure, associativity and the presence of the unit in the set G = {+1; –1; +*i*; –*i*} would be apparent to many pupils. Closure appears clearly both in Bombelli's rules and in the *Cayley table*; the other properties are familiar. The inverse property namely that, for every $x \in G$, there is $x' \in G$ such that $x \cdot x' = x' \cdot x = 1$ can be harder to discern.

Card A. In *Algebra*, Rafael Bombelli of Bologna (1526-1572) gave the rules:

$+1) \cdot (+1) = +1$	$(-1) \cdot (+1) = -1$	$(+1) \cdot (-1) = -1$	$(-1) \cdot (-1) = +1$
$(+1) \cdot (+i) = +i$	$(-1) \cdot (+i) = -i$	$(+1) \cdot (-i) = -i$	$(-1) \cdot (-i) = +i$
$(+i) \cdot (+i) = -1$	$(+i) \cdot (-i) = +1$	$(-i) \cdot (+i) = +1$	$(-i) \cdot (-i) = -1$

(we have written the rules by modern symbols).

Consider the set G = {+1; –1; +*i*; –*i*}. Are the following statements true or false?

(1) The product of two elements of G is always an element of G.
(2) The multiplication of elements of G is associative.
(3) There is an element $e \in G$ such that, for every $x \in G$, $e \cdot x = x \cdot e = x$.
(4) For every $x \in G$, there is an element $x' \in G$ such that $x \cdot x' = x' \cdot x = e$.

Card B. Let us consider the following table:

x	+1	-1	+*i*	-*i*
+1	+1	-1	+*i*	-*i*
-1	-1	+1	-*i*	+*i*
+*i*	+*i*	-*i*	-1	+1
-*i*	-*i*	+*i*	+1	-1

Consider the set G = {+1; –1; +i; –i}. Are the following statements true or false?

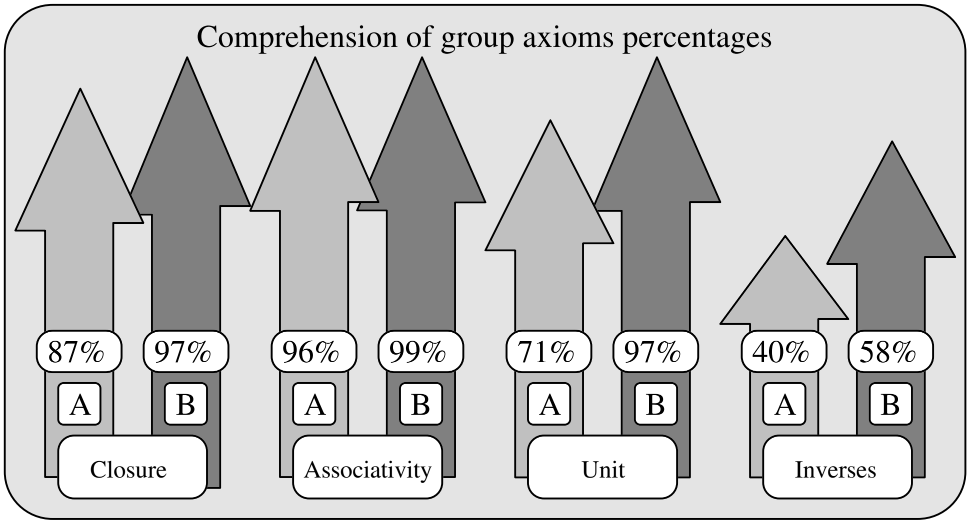
(1) The product of two elements of G is always an element of G.
(2) The multiplication of elements of G is associative.
(3) There is an element $e \in$ G such that, for every $x \in$ G, $e \cdot x = x \cdot e = x$.
(4) For every $x \in$ G, there is an element $x' \in$ G such that $x \cdot x' = x' \cdot x = e$.

The time allowed to read the card and answer the questions was10 minutes. (We wanted students to examine the problem 'at a glance'). The students answers were as follows for the two samples.

Card A	**true**		**false**		**no answers**	
(1)	59	87%	2	3%	7	10%
(2)	65	96%	1	1%	2	3%
(3)	48	71%	3	4%	17	25%
(4)	27	40%	21	31%	20	29%

Card B	**true**		**false**		**no answers**	
(1)	69	97%	2	3%	0	0%
(2)	70	99%	0	0%	1	1%
(3)	69	97%	0	0%	2	3%
(4)	41	58%	8	11%	22	31%

These results indicate that properties 1, 2, 3 (closure, associativity, presence of the unit element) are recognised by students. As regards properties 1 and 2 the differences between sample A and sample B are fairly small, with a greater difference for property 3: it seems that the *Cayley table* was somewhat more helpful to pupils than Bombelli's rules. As regards property 4 (the presence of inverses) the situation is rather different: only 40% of the students (sample A) and 58% (sample B) accepted the property. Let us represent some of the results in the following picture (it is just a qualitative representation: the differences between percentages in test A and in test B are sometimes slight):



Tab. I

We interviewed individually those pupils (A: 41 students; B: 30 students) who did not consider statement 4 to be "true". Almost all these students simply stated that they did not realize the presence of the inverses just by examining Bombelli's rules (or the *Cayley table*). So consideration of the historical example was successful in causing all the conjectured reactions and expected effects for only some of the students (let us notice that it is well known that a multiplicative *finite* submonoid G of the multiplicative group **C*** of non-zero complex numbers is a subgroup: the sufficiency of the closure test, in this case, is underlined also in: Burn, 1996, p. 373; so, as regards the previous tests, it is inconsistent to state that properties 1, 2, 3 are true and property 4 is *not* true; however, high school pupils cannot know the mentioned proposition: Dubinsky & Al., 1997, p. 251).

4. History of mathematics and epistemology of learning

The consideration of relevant examples from the history of mathematics can really help the introduction of important topics. As regards the group concept, however, the supposed reactions took place completely only for *some* students.

A pre-axiomatic start to group theory can be useful (see for example: Jordan & Jordan, 1994), but it is not always enough to assure full learning. As indicated above, the question remains open. It is possible to object that the mere offereing of

Bombelli's rules is insufficient to achieve a complete learning of the group concept. Let us emphasise that our research was an exploratory study. For a fuller investigation it would be necessary to identify clearly sampling criteria and pre-course intuitions (as underlined in: Burn, 1996, p. 371).

Let us quote once again Dubinsky & al. (1997): *"A historical view is useful in designing research and instruction with respect to group theory. History is certainly a part of our methodology, but we are influenced not only by the record of who proved what and when, but also with the mechanisms by which mathematical progress was made*" (Dubinsky & a1., 1997, p. 252; according to Piaget & Garcia, 1983, there is a close connection between historical and individual development at the level of cognitive mechanism; see: Dubinsky & Al., 1997, p. 252).

The main limititation of the notion of mathematics education as *thought transference* lies in the uncertainty about real effects (upon the learning) of teachers' choices. We make no claims for the teaching of abstract algebra – through the consideration of historical references or otherwise – as regards the nature and the meaning of mathematical objects. Here several problems are opened (Godino & Batanero, 1998), involving several fundamental philosophical questions. However it is important and necessary to control the educational research process by experimental verification: this can profoundly affect the delineation of the research and give it an important, particular epistemological status.

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