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Historical roots of limit notion
Development of its representation registers and cognitive development

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Summary. The influence of visualisation and of verbal and symbolic expressions of main infinitesimal methods upon didactics, and in particular its importance for the correct characterisation of concepts, is well known. In this paper different ideas and expressions of infinitesimal methods in the history and in mathematics education are investigated, with particular reference to the limit notion. Historical development of representation registers can lead to a parallel development of the notion in students' minds, and this would make it possible to design new ways to overcome some obstacles and to develop students' ability to use and to co-ordinate different registers; however explaining the problems encountered by mathematicians in history (who inhabited different paradigms with different social knowledge structures and different beliefs) does not necessarily help students with their difficulties. Our main contribution resides in showing that dynamic and static ideas of limit are encompassed by different semiotic registers.

To John Fauvel, 1947-2001

INTRODUCTION

The full discussion of a possible parallelism between history and cognitive growth would require a specific theory of knowledge allowing the comparison of the students' growth of knowledge and the historical development of the concepts; moreover it would be necessary to point out some remarks concerning the effectivity of such parallelism as well as its limitations, mainly connected to different paradigms, with different social knowledge structures and different beliefs, that characterised different stages of the historical development of the concepts. Our main goal is not so high, exacting: it resides in showing that, from the educational point of view, the dynamic and the static ideas of limit, as formulated in different historical stages, are encompassed by different semiotic registers.

In this work we shall not give a summary of educational researches devoted to the limit notion: we shall describe just some elements of the theoretical framework that we are going to consider.

At least since Eighties several studies show that a full understanding of the limit is rather rare ([1]): Schwarzenberger (1980) states that mathematical difficulties connected to classical analysis cannot be simply explained: according to him, an intuitive expression of main infinitesimal ideas implies difficulties for the students. Tall underlines that such difficulties are frequently connected to *mathematical* aspect of analysis, not to *cognitive* aspects: in other words, the limit process is intuitive from the mathematical point of view, but not from the cognitive one (Tall, 1985, p. 51), so sometimes cognitive images conflict with the formal definition of limit. The limit of a function is often considered as a dynamic process (Tall & Vinner, 1981, pp. 156-168), so it is considered in the sense of potential infinity and infinitesimal.

Concerning representation registers ([2]), verbal registers are important for the introduction of infinitesimal concepts, mainly referred to potential infinitesimal; but it is important to underline that a verbal register cannot exist on its own because it depends on the community of practice and on the different meanings that individuals usually give to words and to ideas; so words themselves can lead to doubts and misconceptions. The most common words to communicate infinitesimal notions are: *tends to*, *limit*, *approaches*, *converges*; clearly these words are not equivalent as regard their everyday meanings and students hardly recognise that such expressions have the same mathematical meaning (Cornu, 1980, Davis & Vinner, 1986, pp. 298-300; Monaghan, 1991, pp. 23-24).

Verbal representation registers hardly express the limit concept completely: cognitive conflicts do exist in the learning of limits, so verbal representations can be considered both a limitation and a help for the construction of this concept; and the presence of misconceptions is possible, connected to the use of potential infinitesimal, too. However the direct use of symbolic registers (and of notions of actual infinity and infinitesimal) would be too exacting in the High School: it is difficult for the students to understand the limit concept just by ε-δ definition, and of course a weak understanding of the definition itself can be an obstacle for the full comprehension of the limit notion (Tsamir & Tirosh, 1992) ([3]).

FROM HISTORY TO MATHEMATICS EDUCATION

A. Sfard states that, in order to speak of mathematical objects, it is necessary to make reference to the process of concept formation; and she supposes that an operational conception can be considered before a structural one (Sfard, 1991, p. 10). Concerning the *savoir savant* (a well known Chevallard's expression), the historical development of many notions can be considered as a sequence of stages: an early, intuitive stage, and so on, until the mature stage; and several centuries can pass between these stages. Of course it is necessary to theorise further such first description, mainly in order to overcome a merely evolutionary perspective: such savoir cannot be considered absolutely, so according a traditional classical teleological vision: it must be understood in terms of *cultural institutions*, as Chevallard has pointed out; however a first consideration of the situation may be useful.

In early stages (articulated in several different experiences, to be considered with reference to paradigms available at the times), the focus seems mainly operational; the structural point of view is not a primary one. As we shall see with respect to the limit notion, until Cauchy's work some ideas implicitly connected to actual infinity were not considered. From the educational point of view, a similar situation can be described: in an early stage pupils approach notions by intuition, without a full comprehension of the matter; then their learning becomes better and better, until it can be considered mature.

Is it possible and educationally useful to consider an analogy between these situations, i.e. between historical development and cognitive growth? Is it possible to point out in the passage from the early stage to the mature one, in our pupils' minds, some doubts and reactions that we can find in the passage from the early stages to the mature one as regards the *savoir savant*?

We shall not give general answers to such crucial questions: as noticed in the Introduction, the full discussion of a parallelism between history and cognitive growth would require a specific theory of knowledge ([4]): this is not the goal of our work. However, we must underline that processes of teaching-learning take place nowadays, after the full development of the *savoir savant*; so the *transposition didactique*, whose goal is initially a correct development of intuitive aspects, can be based upon the results achieved in the full development of the *savoir savant*: in fact, pupils' reactions are sometimes similar to reactions noticed in mathematicians in history (Tall & Vinner, 1981) and such correspondence can be an important tool for teachers. But a major issue is related to the correct interpretation of history: as noticed, its use must consider the real evolution of the savoir in terms of cultural institutions. So we shall make reference mainly to the use of different semiotic registers, particularly in order to enhance the students' understanding of static and dynamic ideas of the concept of limit.

HISTORICAL ROOTS: POTENTIAL AND ACTUAL INFINITESIMAL

Concerning the methodology of the historical survey, we are going to propose some examples that can be considered relevant in order to present the evolution of the limit notion, in the sense of the parallel evolution of the cognitive growth, too (although the possibility of such parallel evolution cannot be stated uncritically); our main interest is referred to different representation registers employed: in particular, the passage from the dynamic character of the limit to the static one will be considered.

Historically, both notions of actual and potential infinity are ancient ([5]): Aristotle of Stagira (384-322 b.C.) distinguished actual and potential infinity, but mathematical infinity, in Aristotle's opinion, is merely potential (he, avoiding paradoxes, refused actual infinity: Bostock, 1972-1973). As regards infinitesimal, according to the concept of number line, the ancient conception is only potential, although interesting ideas can be related to the exhaustion argument: in the next paragraph we shall analyse such argument and we shall point out that it can be related to some ideas in the sense of actual infinitesimal; however, as we shall see, proofs by exhaustion argument cannot be considered as real limits ([6]).

The implicit opposition between actual and potential infinitesimal became evident after Calculus' birth, so after works by I. Newton (1642 -1727) and G.W. Leibniz (1646-1716): each of them was mainly responsive to his own primary intuition, which in the case of Newton was physical and in the case of Leibniz algebraic. The importance of notions of actual and potential infinitesimal was remarkable in many researches about Calculus' foundations (Bos, 1975); F. Enriques (1871 -1946) underlined any ambiguity in the Leibnizian concept of differential: derivation is considered by Leibniz as quotient of two *differentiae* or (as named by Jo. Bernoulli and by L. Euler) of two differentials; however, "it is not clear in Leibnizian works if these increments must be interpreted only in potential way, like variable and evanescent quantities, or as actual infinitesimal" (Enriques, 1938, p. 60: we shall reprise later the idea of evanescent quantity in Euler's work: McKinzie & Tuckey, 2001) ([7]). In 20th century, mathematicians reprised some Leibnizian ideas: according to A. Robinson, Leibniz knew by intuition that infinitesimals theory brings to the introduction of ideal numbers that can be considered infinitely small if compared to real numbers. However, neither Leibniz nor his disciples nor his successors gave any rational developments to this idea (Robinson, 1974; see interesting suggestions in: Todorov, 2001).

L. Euler (1707-1783) refused the notion of infinitesimal as quantity lower of any given one and different from zero (Kline, 1972); he was able to distinguish the differential of a function from its increment, but he rarely followed this distinction. He wrote in his *Institutiones Calculi Differentialis*:

"Every quantity can be reduced until it becomes zero and it completely vanishes. But an infinitely small quantity is an evanescent quantity and therefore the thing itself is equal to zero. Moreover this is in keeping with the definition of infinitely small things in which we say that they are lower than any given quantity; surely it would be zero because, if it is not equal to zero, it would be possible to assign to itself an equal quantity, and this is against the hypothesis" (Euler, 1755-1787, quoted in: Kline, 1972).

The idea of evanescent quantities is important; unfortunately Euler did not see the possibility that an evanescent quantity can be a different kind of quantity from a numerical constant. Euler was aware of problems with actual infinitesimals (so he seems to cover himself against objections), but when he actually did mathematics, he preferred a different approach (see the Chapter 7 of Book 1 of Euler's *Introductio in Analysin Infinitorum*: "Du développement des Quantités exponentielles & logarithmiques en Séries": Euler, 1796, 1st edition in French, pp. 84-91; concerning Euler's calculation involving infinite and infinitesimal quantities and its interpretation based upon hyperreals: McKinzie & Tuckey, 2001).

As previously stated, our historical summary does not deal just with concepts: it cannot forget the representations of them, so it must consider the registers in which concepts were (and are) expressed. In order to do that we are going back to one of the most important achievements of Greek mathematics.

EXHAUSTION ARGUMENT AND REPRESENTATION REGISTERS

The attribution of the exhaustion argument is based upon Euclid's *Elements*: for instance, the Proposition 10 of Book 12, which proves that any cone is a third part of the cylinder with the same base and equal height, is attributed to Eudoxus of Cnidus (405-355 b.C.; see: Bourbaki 1963, p. 171; Dieudonné, 1989, pp. 63-64; the term *exhaustion* was used since 17th century: Giusti, 1983, p. 255).

Proofs by exhaustion argument ([8]) are based upon the following proposition:

Proposition 1 of Book 10. Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proved even if the parts subtracted are halves.

In the proof of this proposition, Euclid applies the so-called Eudoxus' postulate:

Definition 4 of Book 5. Magnitudes are said to *have a ratio* to one another which can, when multiplied, exceed one another.

So in *Elements* this postulate is a *definition*: in fact, for instance, the set of rectilinear and curvilinear angles is not a class of Archimedean magnitudes (Proposition 16 of Book 3). And this consideration of curvilinear angles is interesting: *it shows that Greeks were not unaware of quantities that can be infinitesimal* (i.e. quantities that are not real number constants: Tall, 1982).

As above stated, nowadays the exhaustion argument can be considered interesting with reference to representation registers employed, too. First of all, the exhaustion principle can be expressed by *verbal registers*, as Euclid himself did in the quoted Proposition 1 of the Book 10; however we must point out some fundamental remarks. First of all, let us notice that the exhaustion argument is operating at a formal level, whereas "verbal" seems to suggest a more intuitive level. Moreover, as regard verbal registers, and generally all kinds of registers, it is important to underline that there is not a single register of a given kind: in fact the nature of a register depends on the community of practice in question and frequently it is indivisibly linked from other conceptual aspects. For instance, verbal registers make reference to words and to the meanings of those words, meanings that often link to other registers, too (e.g. verbal registers are communicating internally and externally with conscious or inconscious links to spatial, temporal and other senses). So when we make reference to registers, we must always consider, explicitly or implicitly, such dependence on various cultural frameworks.

We can express a proof by exhaustion argument by *visual registers*, too: the proof of the Proposition 2 of Book 12, according to which circles are to one another as the squares on their diameters, can be based upon the visual representation of a circle and of some inscribed and circumscribed similar polygons.

Let's notice moreover that it is possible to express the exhaustion argument by modern *symbolic registers* (see for instance: Carruccio, 1972, p. 167). However, concerning either epistemological or educational issues, we must underline that clearly the exhaustion principle can be referred to an infinitesimal situation; but the question is the following: is it possible to point out a limit (in modern sense) in the exhaustion argument? Several authors do not agree with such statement: so the exhaustion argument cannot be considered equivalent to a real limit (Kline, 1972, pp. 99-100) ([9]). And the non-equivalence is not only in the formal sense: the most important differences pertain to the ontological realm (Radford, 1997). In our opinion, a direct comparison between a proof by exhaustion argument and a modern limit would be historically and epistemologically weak, nearly meaningless: while "*internalist* history of mathematics (...) tends to interpret the past in terms of modern concepts, more recently researchers have tried to take a more holistic view, with mathematics seen as a component of the contemporary culture; the historian's task is then to discover the influences, conditions and motivations (...) under which problems arose" (Grugnetti & Rogers, 2000, p. 40). We completely agree.

THE LIMIT IN WALLIS, IN CAUCHY, IN WEIERSTRASS

Historical roots of the *limit* notion are not as ancient as historical roots of infinitesimal methods (Rufini, 1926). J. Wallis (1616-1703), in his *Arithmetica infinitorum* (1655), introduced an arithmetical concept of the limit of a function, i.e. the number whose difference from the function can be lower than any given quantity. M. Kline underlines that "its formulation is still vague, but there is the correct idea" (Kline, 1972; as regards this Wallis' *vague formulation*, we shall reprise the important question of correctness at the end of this paragraph). Another mathematician, in 17th century, worked about the limit: according to G. Loria, in *Elementum tertium* from *Geometriae speciosae elementa* (1659) by P. Mengoli (1635-1686), "this mathematician showed to get a clear idea of the limit concept" (Loria, 1929-1933, p. 526). Some interesting notes are in *Vera circuli et hyperbolae quadratura* (1667) by J. Gregory (1638-1675) and in Newton's masterpiece, *Philosophiae naturalis principia mathematica* (1687; moreover: Castelnuovo, 1938; Boyer, 1969; Menghini, 1982; Edwards, 1994).

We must underline that these limit notions were sometimes mainly related to sequences and to series: F. Viète (1540-1603), in his *Varia responsa* (1593), calculated the sum of a geometric series; P. de Fermat (1601-1665), too, knew this result; in 1655 A. Tacquet (1612-1660) and J. Wallis published it in *Arithmeticae theoria et praxis accurate demonstrata* and in *Arithmetica infinitorum*. In particular, Tacquet underlined that the passage from a "finite progression" to an infinite series is "immediate" (Loria, 1929-1933, p. 517). Gregory of St. Vincent (1584-1667) in his *Opus geometricum* (1647) referred the paradox of Achilles and the Turtle to a geometric series and wrote:

"The conclusion of a progression is the end of the series that the considered progression does not reach, although it is indefinitely lengthened; it can approach such value as close as it is possible" (quoted in: Kline, 1972).

Let us now consider an interesting educational reference: Gregory's comment illustrates a major misconception in limits. He refers to a sequence whose terms are always different from the limit (about limits of sequences and sums of series, see: Boyer, 1969 and 1982); Tall and Vinner underlined that frequently students think that $s_n \to l$ means that values of the sequence s_n just *approach* the limit l, but never reach it (a nice popular example is whether 0.999... is equal or less than 1; we examined it in: Bagni, 1998); in this situation, the limit of a function is clearly considered as a dynamic process (Tall & Vinner, 1981, pp. 156-168), so it is considered in the sense of potential infinity and infinitesimal. ([10])

G. Vitali (1875-1932) noticed that "of course the convergence could not be considered before the limit notion" (Vitali, 1979, p. 404) and the limit was not correctly considered as the fundamental analytical concept (in particular, in 17th and in 18th centuries, the question of the existence of the limit of the sequence of partial sums was not considered): however we can state that the limit process is met before the limit concept.

According a traditional classical vision, A.L. Cauchy (1789-1857) was the first mathematician to make a *rigorous* study of the Calculus ([11]). In our opinion, correctness must be always investigated in its own conceptual context and not against contemporary standards, in order to avoid the imposition of modern conceptual frameworks to works based upon different ones (so Euclid and Wallis *were* rigorous *in their own ways*). However Cauchy's *Cours d'Analyse algébrique* (Paris, 1821), a book particularly designed for students at École Polytechnique, must be considered a fundamental treatise from the formal point of view, too, and it developed many basic analytical theorems as rigorously as possible.

Let us remember Cauchy's definition of limit and of infinitesimal (from *Cours d'analyse*, p. 4: Cauchy, 1884-1897):

"When values of a variable approach indefinitely a fixed value, as close as we want, this is the *limit* of all those values. For instance, an irrational number is the limit of the different fractions that gave approximate values of it (…). When values of a variable are (…) lower than any given number, this variable is an *infinitesimal* or an infinitesimal magnitude. The limit of such variable is zero" (quoted in: Bottazzini, Freguglia & Toti Rigatelli, 1992, pp. 327-328; as regard original texts, see: Smith, 1959).

So Cauchy introduced the fundamental distinction between *constants* and *variable quantities*, although he had no formal description of real numbers as ordered fields satisfying a list of axioms.

Would it be possible to express Cauchy's verbal definition by symbolic registers? We shall not give an answer: "Mathematics is not just text; it lives in the minds of people and can, to an extent, be disclosed by interpreting the artefacts they have produced" and "these artefacts, inscriptions, instruments, books and technical devices, have been developed in particular places for particular reasons" (Grugnetti & Rogers, 2000, p. 46). Cauchy's formulation was expressed in the paradigm available at the time, and its formulation can lead to the use of different registers: Tall notes that "there are several ways of visualising infinitesimals as points of as line", which allow "the novice to *see* infinitesimals as *variable points* that are *arbitrarily smaller* than any positive real constant c. This is analogous to the notion prevalent at the beginning of the 19^{th} century when Cauchy described infinitesimals as *variable quantities that tend to zero*" (Tall, 2001, p. 225).

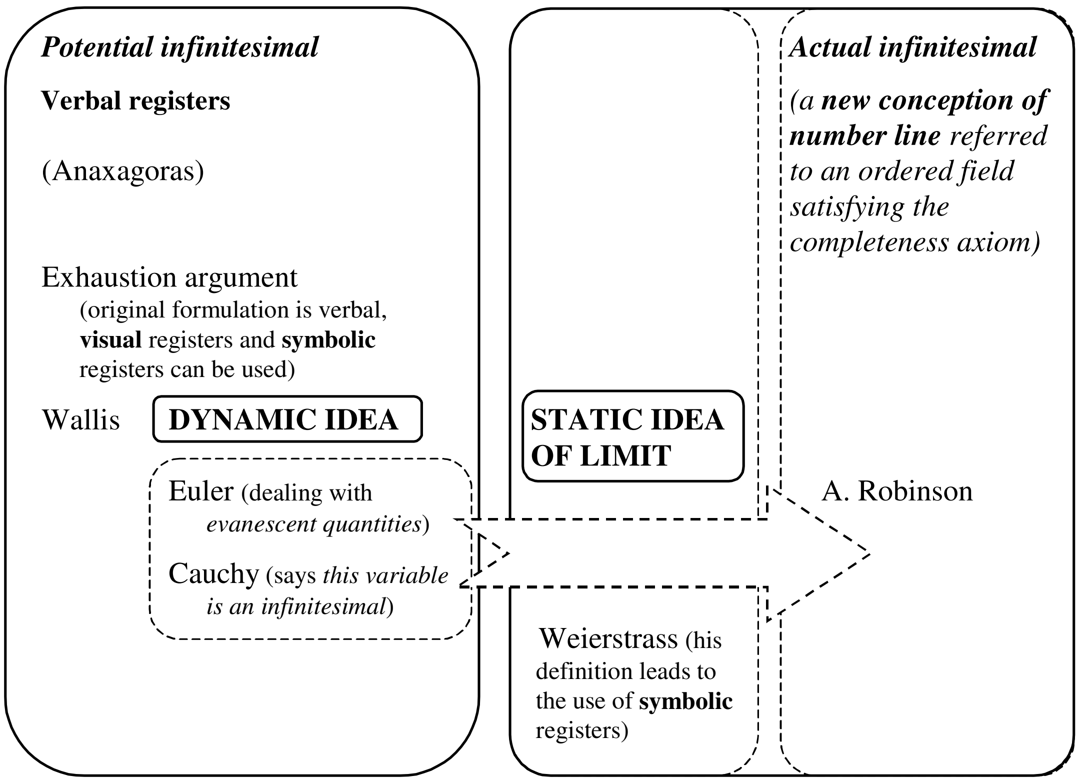
So the modern limit notion, pointed out since 17^{th} century by Wallis, Mengoli and, finally, by Cauchy, is mainly expressed by verbal representations (although it is important to remember that such verbal registers cannot be isolated, for instance, from the sensory perception of *arbitrarily small things*).

K.T.W. Weierstrass (1815-1897) gave the modern definition of limit and of continuous function: in fact he stated that the function $x \rightarrow f(x)$ is continuous in $x = c$ if, for any real number $\varepsilon>0$, we can find a real number $\delta>0$ such that for every x such that $|x-c|<\delta$ we have $|f(x)-f(c)|<\varepsilon$. Let us quote M. Kline, who noticed that "Weierstrass' work improved previous works by Bolzano, Abel and Cauchy. Weierstrass tried to avoid intuition (...) and did not like the sentence *a variable approaches a limit* because it would suggest ideas of time and motion" (Kline, 1972).

With respect to representation registers, we must notice that Weierstrass' conception and definition of limit really allow a modern symbolic representation: "for every $\varepsilon>0$ there is $\delta>0$ such that for every x such that $|x-c|<\delta$ being $x \neq c$ we have $|f(x)-l)|<\varepsilon$" can be considered quite equivalent to Weierstrass' definition of the limit l (and it can be expressed using quantifiers, by symbols like $\forall$, $\exists$ etc.): so we conclude that Weierstrass' definition (the so-called ε-δ definition) finally leads to the use of symbolic representation registers. However, let us underline once again that it would be misleading to make reference to a single symbolic representation register: there are different symbolic registers in different communities of practice. Leibniz, Newton, Cauchy, Robinson had their own symbolic registers which differ from each other and of course differ, too, from that of Weierstrass.

From the educational point of view, the main difficulty for students dealing with the ε-δ definition is the static character of the formal theory versus the dynamic character of the cognitive approach: the consideration of historical development of the concept of limit, mainly with reference to the use of different semiotic registers, can be useful in order to make possible the correct formulation of the static and the dynamic ideas of limit.

Let us summarise in the following table some remarks previously pointed out.



FINAL REFLECTIONS

In our opinion, the use of representation registers as a tool to analyse both historical and educational aspect of the limit notion can be an interesting track to follow, although the question to be considered is that it is not completely clear how philogenetic processes are related to ontogenetic ones. Moreover, from the educational point of view, experimental verifies would be necessary: many aspects influence the learning of infinitesimal concepts, e.g. some clauses of didactic contract; those influences can be pointed out by tests and interviews; of course, concerning experimental aspects, it would be necessary to identify sampling criteria and, for instance, pre-course intuitions (and to consider the experimental contract). In this work we don't propose experimental verifies: so we don't give now general results and conclusions, but we just suggest some final reflections.

'The limit notion occurs in a number of different guises. (...) All of these have in common a *process* of getting arbitrarily close to a fixed value (the limit). In every case, the same symbolism is used both for the process of convergence and also for the *concept* of limit" (Tall, 2001, p. 232; Tall & Al., 2001): the historical development allows us to consider different approaches, related to potential or to actual infinitesimal and to different representation registers; however, from the educational point of view it is really difficult to introduce the limit notion without making reference to the limit *procept* (Gray & Tall, 1994).

In our opinion, the problem of the passage from discrete of continuum is mainly a cultural one, and historical issues are important in order to approach it and to overcome many difficulties (Radford, 1997; Furinghetti & Radford, 2002) ([12]). Further researches can be devoted to clarify what category of persons should be acting on above considered argument and in what way: the remarks about knowledge of the possible parallelism between use of registers and historical development are addressed towards students, teachers, mathematics educators, researchers in mathematics education. Surely several questions are still opened: there is a *possible* parallelism between the historical development of ideas about infinitesimals and the developments within a student's understanding; what follows? For instance, the student will be really helped by knowing more of the history? What about the important reading of primary sources? And what is teacher's role? Is it the mathematics educator responsible for training the teachers who should be aware of this parallelism and how does this help in teacher-training? ([13])

We propose a final reflection: using examples from history of mathematics can be effective to introduce some fundamental topics, e.g. the static and the dinamic ideas of limit with reference to semiotic registers employed; it allows interesting a-priori analysis of the difficulties of the students and it makes it possible to design new ways to overcome classical obstacles. However explaining the subtle problems encountered by savants in history does not *necessarily* help students with their difficulties, since mathematicians in history just inhabited different paradigms with different social knowledge structures and different beliefs. So using examples from history must be controlled in order to obtain full learning, for instance by evaluating empirical details of the work with students.

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Notes

([1]) Several works were devoted to didactics of the limit: A. Sierpinska states that obstacles can be classified into five groups (Sierpinska, 1985): in the first one, named *Horror infiniti*, we find the refusal of the status of mathematical operation for the limit and the consideration of a concrete approaching; then Sierpinska considers obstacles connected to the function concept, to the geometric interpretation of limits, to logic and to symbology (Sierpinska based her work upon the topological conception of limit, while, for instance, B. Cornu's research is based upon the numerical one: Cornu, 1991; Artigue, 1998). A lot of references can be quoted: Tall & Vinner, 1981; Cornu, 1980, 1981; Orton, 1983; Davis & Vinner, 1986; Sierpinska, 1987; Mamona, 1987; Monaghan, 1991; Tall, 1990a, 1990b, 1994; as regard summaries: Mamona-Downs, 1990, Gagatsis & Dimarakis, 1996.

([2]) As regards a definition of representation registers, see: Duval, 1995. See moreover: Kaput, 1991; stimulating suggestions can be found in: Thurston, 1994.

([3]) We just notice that we cannot consider didactics of Calculus only with reference to the limit notion: for instance, concerning the concept of derivative, we remember the primary role played by cognitive roots (the *local straightness* is the cognitive root with reference to derivative: Tall, McGowen & DeMarois, 2000); this remark deals with the general function concept. Many researches (Trouche, 1996, based upon: Rabardel, 1995; Artigue & Al., 1997) underline that the use of technological media induces behaviours based upon the comparison of different points of view, so it requires a remarkable ability to co-ordinate different semiotic registers (Confrey, 1992, Dubinsky, 1995, Nemirovsky & Noble, 1997; Yerushalmy, 1997; see summaries in: Tall, 1996; Artigue, 1998).

([4]) It seems to bear out a famous statement by Piaget and Garcia, according to which historical and individual development are linked (see: Piaget & Garcia, 1983; of course, by that, we do not support this idea by Piaget and Garcia in *all* educational situations). Such considerations are supported by some classical works by Chevallard (1985) and Sfard (1991; concerning Sfard's work, let's underline that she tackles the problem by using Piaget's epistemology); however, we don't concentrate on the discussion about eventual problems of co-ordinating results from different theoretical frameworks.

([5]) Anaxagoras of Clazomenae (500?-428 b.C.) wrote: "For neither is there a least of what is small, but there is always a less. For being isn't non-being" (quoted in Geymonat, 1970, I). If we consider that underlying the concept of limit (but independent of it) there is the concept of the number system as a continuum and an Archimedean ordered field, Anaxagoras seems to be addressing the number system rather than limit directly; and it would be necessary to consider the difference between magnitude and number in Greek contribution, mainly with reference to the different ways in which a geometric object might be conceived (see the two definition of proportion in Euclid: Def. 7 of Book 5 and Def. 21 of Book 7). The concept of number line is quite different as seen by Greeks, Cauchy or Robinson; the insight of modern theories lies in the possibility of considering systems having different kinds of quantity (for instance, Cauchy distinguished between constants and variable quantities: in his own opinion, infinitesimals are quantities but not constants; however such distinction between constants and variable quantities cannot be considered until the introduction of modern arguments: Tall, 1982).

([6]) Results to be proved by *reductio ad absurdum* must be known by intuition or by heuristic techniques that were not accepted by Greeks as full proofs.

([7]) Leibniz wrote (1695): "When we mention quantities (…) indefinitely small (the lowest we can know) we mean that we want signify quantities (…) as small as we want, so the mistake if we should make is lower than any given quantity" (quoted in: Kline, 1972). Some notes written by Leibniz to Wallis in 1690 are interesting: "It is useful to consider quantities infinitely small such that, when we look for their quotient, they cannot be considered equal to zero, but that are refused when they appear together greater quantities. So, if we have $x+dx$, dx is refused. But the situation is quite different if we look for the difference between $x+dx$ and x. In a similar way, we cannot consider xdx and $dxdx$ together. So if we must differentiate xy, we write $(x+dx)(y+dy)-xy = xdy+ydx+dxdy$. So, in every particular situation, the mistake is lower than every finite quantity" (Leibniz, 1849-1863, IV, p. 63). Infinitesimal methods had, in 18th century, some active opponents, like G. Berkeley (1685-1753), who wrote the pamphlet *The Analyst, or a discourse addressed to an infidel mathematician*. He stated: "I confess that the notion of a quantity infinitely lower than every sensible or imaginable quantity goes beyond my capability. But the notion of a part of this infinitesimal quantity such that it is still infinitely lower than itself, this is an infinite difficulty for every man" (Arrigo & D'Amore, 1992, p. 123). M. Rolle (1652-1719) himself expressed doubts about the settlement of infinitesimal notions: he was not certain about the correctness of Leibnizian Calculus, that "he considered as a sort of successful trick" (Bottazzini, 1990, p. 27). J.-B. d'Alembert (1717-1783) stated: "A quantity is something or it is nothing: if it is something, it did not become zero yet; if it is nothing, it really became zero. The supposition that there is an intermediate state between something and nothing is a wild fancy" (*Mèlanges de litèrature, d'histoire et de philosophie*, p. 249, quoted in: Boyer, 1982).

([8]) The principle of exhaustion is used in many propositions in Book 12 of *Elements*: main results are Prop. 2, 5, 10, 18.

([9]) For instance, G. Saccheri (1667-1733) in his *Euclides ab omni naevo vindicatus*, wrote about the Prop. 2: "Euclid previously proved (Prop. 1) that similar polygons inscribed in circles are to one another as the squares on their diameters; then he would deduce the Prop. 2 by considering circles as polygons with infinitely many sides" (Saccheri, 1904, p. 104). But Euclid (and Eudoxus, too) *never* used infinity according to this idea (Euclid, 1970, p. 931).

([10]) The question of whether a sequence can reach its limit or not can be considered a philosophical one: some sequences (e.g. constant sequences) really do reach the limit; the main question is whether the prototypical sequences getting *closer and closer* do reach the limit *at infinity* in some sense: in fact, this potential process may never reach its limit; and once again this brings to a cognitive conflict, where cognitive images clash with the mathematical formal definition. Moreover it is important to underline that each representation register actually has a different measure of *closeness* (see D. Tall's *Graphic Calculus*: Tall, 1986; as regard inconsistencies: Tall, 1990b).

([11]) Let us just quote some words by the young N.H. Abel (1802-1829): "Cauchy is a fool", but he is the only man "who knows the real way to do mathematics" and "who nowadays deals with pure mathematics" (Bottazzini, 1990, p. 86).

([12]) Concerning some avenues of research, we remember that Lakoff and Nuñez (2000) argue that conceptual metaphor plays a central role into mathematical ideas like infinity and infinitesimal (see for instance the *Basic Metaphor of Infinity*; of course their work deals with many other mathematical concepts, too). The fundamental work by Lakoff and Nuñez is mainly devoted to cognitive aspects: embodiment is one of the most important issues of the research in mathematics education and it is important to investigate several subtle connections between perceptions and symbols; however from a strictly epistemological point of view, the crucial point is the passage from discrete to continuum; and metaphorical reasoning, very important from the educational point of view, must be controlled by the teacher in order to avoid misguided generalisations (let us remember, for instance, difficulties with non-convergent series; cases of overgeneralization are discussed in: Bagni, 2000).

([13]) Concerning teachers, a relevant epistemological skill is needed! Suggestions can be found, for instance, in: Weil, 1980; Swetz, 1982, 1989, 1992, 1995; Katz, 1986; Fauvel, 1990; Anglin, 1992, Marchisotto, 1993, Nobre, 1994, Jahnke, 1995, Siu, 1995, Calinger, 1996, Furinghetti & Somaglia, 1997.

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