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Irrational inequalities: learning and *didactical contract*

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Summary. In this paper we examinate by a test the learning of an important "chapter" of the mathematical education in the High School (in particular, referred to pupils aged 16-17 years, of II and III classes of the Italian *Liceo scientifico*). Many students are conditioned from *didactical contract* to apply always the "standard" rules given by the teacher, and this behaviour can be harmful: resolutions are mechanical, not creative and sometimes they become more complicated.

The influence of the *didactical contract* on the students is remarkable: in this paper, we shall examinate a "chapter" of the mathematical education in the High School (in particular, pupils aged 16-17 years) and we shall expose some considerations about its learning [Brousseau, 1987].

Irrational inequalities are often introduced (for example in the II class or in the III class of the Italian *Liceo scientifico*) particularly referring to the irrational inequalities of 2nd degree. Their resolution is based upon rules, given by the teacher and often learnt by heart by the students; but not always, as we shall see, their application is critical.

The resolution of an irrational inequality is a very common exercise, in the standard mathematical curriculum of the High School. The following rules are well-known:

$$\sqrt{\mathrm{A}(x)} \le \mathrm{B}(x) \qquad \Rightarrow \qquad \begin{cases} \mathrm{A}(x) \ge 0 \\ \mathrm{B}(x) \ge 0 \\ \mathrm{A}(x) \le \left[\mathrm{B}(x)\right]^2 \end{cases}$$

$$\sqrt{\mathrm{A}(x)} \ge \mathrm{B}(x) \qquad \Rightarrow \qquad \begin{cases} \mathrm{A}(x) \ge 0 \\ \mathrm{B}(x) \le 0 \end{cases} \vee \begin{cases} \mathrm{B}(x) \ge 0 \\ \mathrm{A}(x) \ge \left[\mathrm{B}(x)\right]^2 \end{cases}$$

The resolution of *any* irrational quadratic inequality is often based upon the (critical or mechanical?) application of these rules. First of all, let us underline that this application is *not always* strictly necessary; let us consider, for example, the inequality:

$$\sqrt{x} + 2 \le 0$$

It is verified by no $x \in \mathbf{R}$: but it is *not* necessary to use the rules to obtain this result. Their use brings to the following (correct) resolution:

$$\sqrt{x} + 2 \le 0 \quad \Rightarrow \quad \sqrt{x} \le -2 \quad \Rightarrow \quad \begin{cases} x \ge 0 \\ -2 \ge 0 \\ x \le (-2)^2 \end{cases}$$

and this is verified by no $x \in \mathbf{R}$.

This application makes the resolution more difficult: it would be easy to note that $\sqrt{x}$ cannot be negative, so the sum $\sqrt{x} + 2$ is of course positive...

But the habit to use rules, formulas to solve exercises and problems brings sometimes the students to a mechanical approach; several students seem say: to solve this exercise (an irrational inequality of 2nd degree), the teacher gave one or more rules, one or more formulas, so *every* exercise like it (*every* irrational inequality of 2nd degree) *must* be solved by those rules and by those formulas [D'Amore-Sandri, 1993].

METHOD OF TEST

The following test was proposed to students belonging to two 3rd classes of a *Liceo scientifico* (High School) in Treviso, Italy, total 52 students (their mathematical curricula were standard; they knew the rules and the formulas to solve irrational inequalities of 2nd degree):

Solve in $x \in \mathbf{R}$:			
	a) $\sqrt{x} \le 6 - x$	*b)* $\sqrt{x-1} \ge 1 - x$	*c)* $\sqrt{x} \le 1$
	d) $\sqrt{x} \ge 0$	*e)* $\sqrt{x} \le -1$	*f)* $\sqrt{-x} \ge 0$

(Time: 30 minutes; scientific calculator not allowed).

Questions (*a*) and (*b*) need the remembered rules; the resolution of the question (*c*) can be achieved with or without those rules. Resolutions of questions (*d*), (*e*), (*f*) are possible without the rules, and their application is sometimes harmful: the resolution becomes more complicated.

RESULTS OF TEST

QUESTION (a): $\sqrt{x} \leq 6 - x$.

Total.	Correct answers:	34	(66%);
	Correct but not complete answers:	7	(13%);
	Uncorrect answers:	9	(17%);
	No answer:	2	(6%);

QUESTION (b): $\sqrt{x-1} \geq 1 - x$.

Total.	Correct answers:	30	(58%);
	Correct but not complete answers:	8	(15%);
	Uncorrect answers:	10	(19%);
	No answer:	4	(8%);

QUESTION (c): $\sqrt{x} \leq 1$.

Total.	Correct answers:	36	(69%);
	Correct but not complete answers:	4	(8%);
	Uncorrect answers:	10	(19%);
	No answer:	2	(4%);

QUESTION (d): $\sqrt{x} \geq 0$.

Total.	Correct answers:	31	(60%);
	Correct but not complete answers:	1	(2%);
	Uncorrect answers:	19	(36%);
	No answer:	1	(2%);

QUESTION (e): $\sqrt{x} \leq -1$.

Total.	Correct answers:	31	(60%);
	Correct but not complete answers:	3	(6%);
	Uncorrect answers:	10	(19%);
	No answer:	8	(15%);

QUESTION (f): $\sqrt{-x} \geq 0$.

Total.	Correct answers:	32	(61%);
	Correct but not complete answers:	1	(2%);
	Uncorrect answers:	16	(31%);

No answer:	3	(6%).

Notice that 58% of inequalities of the questions (*d*), (*e*), (*f*) are solved by formulas remembered; this 58% can be divided: 36% are correct or correct but not complete, 22% are uncorrect.

CONSIDERATIONS ABOUT RESULTS

- The students are skilful enough in solving inequalities like (*a*), (*b*), (*c*) by the "standard" rules (failures are often caused by miscalculations); but several pupils dislike exercises whose resolution is *not* based upon the rules.

For the questions (*a*), (*b*), (*c*), we have in total:

Correct or correct but not complete answers:	119	(76%)
Uncorrect answers or no answers:	37	(24%)

For the questions (*d*), (*e*), (*f*), we have in total:

Correct or correct but not complete answers:	99	(63%)
Uncorrect answers or no answers:	57	(37%)

So the total ratio of failures is higher (37% versus 24%, although the number of the stdents is low) for the inequalities of the second group.

- In spite of their easiness, exercises (*d*), (*e*), (*f*) are often set off (58%) and solved, sometimes correctly (35%), applying the "standard" rules; for example (question *d*):

$$\sqrt{x} \geq 0 \qquad \Rightarrow \qquad \begin{cases} x \geq 0 \\ 0 \leq 0 \end{cases} \vee \begin{cases} 0 \geq 0 \\ x \geq 0^2 \end{cases} \qquad \Rightarrow \qquad x \geq 0$$

Sometimes, students try to solve some inequalities without the "rules" (however their use is preferred by the greater number of the pupils). For example, for the questions (*d*), (*e*), (*f*), the total ratio of correct or correct but not complete answers is 63% (total 99 exercises) and we can note:

Correct or correct but not complete answers with "standard" resolution:	56	(36%)
Correct or correct but not complete answers without "standard" resolution:	43	(27%)

- Not always a non-"standard" approach is successful. For example, the following mistake is frequent:

$\sqrt{x} \geq 0 \Rightarrow \forall x \in \mathbf{R}$	(18 times)	(question *d*)

Moreover the following mistakes are interesting:

$\sqrt{x} \leq -1 \Rightarrow 0 \leq x \leq 1$	(6 times)	(question *e*)
$\sqrt{-x} \geq 0 \Rightarrow \varnothing$	(6 times)	(question *f*)
$\sqrt{-x} \geq 0 \Rightarrow \forall x \in \mathbf{R}$	(9 times)	(question *f*)

JUSTIFICATIONS GIVEN BY STUDENTS

Several students gave interesting justifications, particularly referred to the answers to the questions (*d*), (*e*), (*f*) ("formulas" are the "standard" rules above remembered).

Mistake: $\sqrt{x} \geq 0 \Rightarrow \forall x \in \mathbf{R}$ (question *d*)

Valentina says: "I forgot the condition for the reality of the root", and many students agree. Francesco says: "I did not use formulas so I did not think I had to put the condition for the reality of the root. Well, formulas are better!"

Mistake: $\sqrt{x} \leq -1 \Rightarrow 0 \leq x \leq 1$ (question *e*)

Luca's remark is interesting: "I thought that $\sqrt{x} \leq -1$ and $\sqrt{x} \leq 1$ were... the same exercise, because I know that $(-1)^2 = 1$".

Mistake: $\sqrt{-x} \geq 0 \Rightarrow \varnothing$ (question *f*)

Enrico says: "I saw the $-x$ under the square root: impossible! I did not think that if x is negative then $-x$ is positive".

Mistake: $\sqrt{-x} \geq 0 \Rightarrow \forall x \in \mathbf{R}$ (question *f*)

This mistake, too, is connected to the condition for the reality of the root. Valentina's remark is interesting: "I knew that a square root is always positive [this was her first statement; later she said: "not negative"], and I forgot the condition for the reality of the root: I made the same mistake in $\sqrt{x} \geq 0$. And... two different inequalities cannot have the same result!".

CONCLUSIONS

- The application of the "rules" given by the teacher in the questions (*a*), (*b*), (*c*) is often correct: inequalities whose resolution needs the rules are often well solved (76% correct or correct but not complete answers). But these rules are frequently applied to very simple exercises, too, and this is unuseful, sometimes harmful (resolutions become more complicated).

Why students do that?

In our opinion, several students are skilful in solving "standard" irrational inequalities of 2nd degree (like $\sqrt{A(x)} \le B(x)$ and $\sqrt{A(x)} \ge B(x)$, with A(*x*), B(*x*) noncostant polynomials in *x*), *employing some fixed rules*; so this skill conditions them. They know that they can solve many given exercises by "standard" rules given by the teacher: so, by that, they surely will satisfy the teacher; and the evaluation of their tests will be positive (importance of the *didactical contract*) [Brousseau, 1987] [D'Amore, 1993].

In students' minds:

resolution of an irrational quadratic inequality	$\Leftrightarrow$	*application of the rules given by the teacher*

So many students are conditioned (from the *didactical contract*) to *apply the rules given by the teacher*, also in very easy exercises (for example $\sqrt{x} \ge 0$, question *d*).

- When the resolution is not "standard", some mistakes are frequent and they are caused by the absence of the condition for the reality of the root:

$\sqrt{x} \ge 0 \Rightarrow \forall x \in \mathbf{R}$ (question *d*)

$\sqrt{-x} \ge 0 \Rightarrow \forall x \in \mathbf{R}$ (question *f*)

Frequently the condition for the reality of the root is wrong; $-x$ is considered a negative number (so it *cannot* be placed... under a square root!):

$\sqrt{-x} \ge 0 \Rightarrow \varnothing$ (question *f*)

It seems that "standard" rules (correctly applied in "standard" exercises) *are not entirely learnt by the students*: the condition for the reality of the root, for example, is considered just one of the parts of the given rule (remember Francesco's remark).

- Let us underline the importance of the teacher: *didactical contract is not chosen by the students*; a pupil can be induced to use always (and, sometimes, mechanically) rules and formulas by teacher's behaviour [D'Amore, 1993] [Wertheimer, 1959]. Of course, a creative behaviour of the students is strictly based upon a creative, open, stimulatig behaviour of the teacher.

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