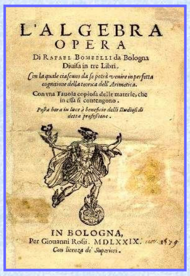
CERME–6 Lyon, 30 Jan. 2009

Working Group 4 – Algebraic Thinking

Rafael Bombelli's Algebra (1572) and a new mathematical "object": a semiotic analisys



To Alberto Favali, 1925–2008

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The introduction of imaginary numbers: a "mistake"?

- The introduction of imaginary numbers is an important step of the mathematical curriculum.
- In the (Italian) Middle School, the impossibility to calculate the square root of negative numbers is frequently reminded to pupils. Later pupils themselves are asked **to consider a new "mathematical object", the square root of –1**, named *i*, and of course this can cause perplexity in our students' minds.
- We shall consider an educational approach based upon an historical reference (*Algebra* by Rafael Bombelli, 1572) that can help us to overcome aforementioned difficulties.

An equation solved according to Cardan and Bombelli

- The resolution of the equation (in modern notation):

$$x^3 = 15x+4$$

is based upon $(q/2)^2–(p/3)^3 = –121$ and leads to the sum of radicals:

$$x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$$

- Taking into account that:

$$2+11i = (2+i)^3 \qquad \text{and} \qquad 2–11i = (2–i)^3$$

we can conclude that **a (real) solution** of the given equation is:

$$x = (2+i) + (2–i) = 4$$

An equation solved according to Cardan and Bombelli

- In the following image we propose **the original resolution** on p. 294 of Bombelli's *Algebra*:

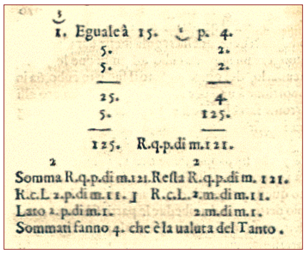
$x^3 = 15x+4$

$x^3 = px+q$

$(q/2)^2–(p/3)^3 = –121$

$x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$

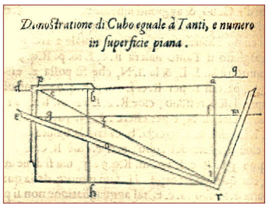
$x = (2+i) + (2–i) = 4$

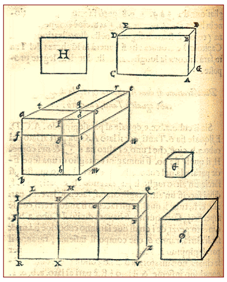


1. Egualeà 15. p. 4.
5. 2.
5. 2.
25. 4
5. 125.
125. R.q.p.di m.121.
Somma R.q.p.di m.121. Resta R.q.p.di m. 121.
R.c.L.2.p.di m.11. R.c.L.2.m.di m.11.
Lato 2.p.di m.1. 2.m.di m.1.
Sommati fanno 4. che è la ualuta del Tanto.

An equation solved according to Cardan and Bombelli

- Let us report the **geometrical constructions**, both two– and three–dimensional, that Bombelli proposed to confirm his procedure (*Algebra*, pp. 296 and 298).





From history to mathematics education

- So the historical introduction of imaginary numbers, did not take place in the context of *quadratic* equations, as in $x^2 = –1$. **It was based upon the resolution of *cubic* equations, whose consideration can be advantageous.**
- In fact, their resolution sometimes does not take place entirely in the set of real numbers (but one of their results is always real). A substitution of $x = 4$ in the equation above ($4^3 = 15·4+4$) is possible in the set of real numbers. In $x^2 = –1$, the role of i is important: results are not real, so their acceptance needs the knowledge of imaginary numbers **(didactic contract).**

A Peircean approach: the *unlimited semiosis*

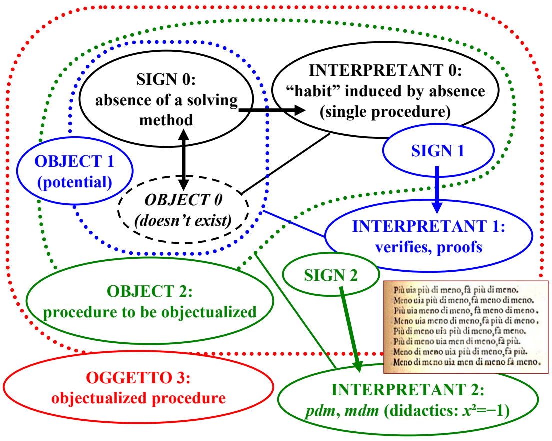
- Now we shall consider some features of students' approach, making reference, in doing so, to Peirce's *unlimited semiosis*.
- According to Peirce, every step of the interpretative process produces a new "interpretant *n*" that can be considered the "sign *n*+1" linked with the object
- **What about the very first sign to be associated to our "object"?**

A Peircean approach: the *unlimited semiosis*

- **"Absence" itself can be considered as a sign.**
- Peirce (*Collected Papers*, § 5.480) made reference to «a strong, but more or less vague, sense of need» leading to «the *first logical interpretants* of the phenomena».
- So we can suppose that this kind of absence can be **the starting point of the semiosic process.**

A Peircean approach: the *unlimited semiosis*

- This is influenced by many elements, *e.g.* **the persons** (pupils, teacher), **the social and cultural context –** we don't state that ontogenesis recapitulates phylogenesis!
- Our starting point can be described as a complexus of "object–sign–interpretant" without a particular "chronological" order: a *habit* linked to the absence of a procedure, *a procedure to be objectualized*.
- Later, with **the emergence of formal aspects,** our object will become "rigorous" (making reference, of course, to the conception of rigor in an historical and cultural context – the rigor for Bombelli and the rigor for modern mathematicians are different).



The root of semiotic chain

- According to V. Font, J.D. Godino and B. D'Amore (2007, p. 14), «what there is, is a complex system of practices in which each one of the different object/representation pairs […] permits a subset of practices of the set of practices that are considered as the meaning of the object».
- Further research can be devoted to the analysis of these practices. Nevertheless the semiosic chain can hardly be considered in the sense of semiotic function. **It can be considered as a first practice that will be followed by other practices in order to constitute the meaning of the "mathematical object".**

The root of semiotic chain

- Of course these reflections cannot be considered in a definitive sense…
- So let us finally quote **Hans–Georg Gadamer** who in 1972 wrote (in his *Truth and Method*):

«It would be a poor hermeneuticist who thought he could have, or had to have, the last word»