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Chapter 3

Integrating history: research perspectives

3.6 Difficulties with series in history and in the classroom

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The history of mathematics provides a collection of useful examples for assisting in the learning of mathematics, which can be used by the teacher in a number of ways (Fauvel 1990, 1991; Pepe 1990; Barbin 1991; Grugnetti 1992; Furinghetti 1993; Furinghetti and Somaglia 1997). This section examines some topics in the history of infinite series which help us to understand better the difficulties faced by today's pupils.

The study of infinite series is an important topic of the mathematical curriculum of the upper secondary school. For several centuries it has played a central role in the study of analysis (on which see Boyer 1969; Edwards 1994), as well as provided a number or counter-intuitive obstacles for the learner. A sum of infinitely many addends, for example, is often considered by pupils to be 'infinitely great'. In this instance the history of mathematics can both help the teacher to understand the pupil's difficulty and suggest what to do about it.

A time-honoured problem in this area is Zeno of Elea's paradox of 'Achilles and the Tortoise'. This concerns a convergent geometric series. Pupils may experience difficulty in absorbing the difference between convergent, divergent and indeterminate series, and this can cause problems and inconsistencies in their minds. Let us consider directly a famous indeterminate series, the one consisting of +l and −1 in alternation. In 1703, the Italian mathematician Guido Grandi stated: "From 1−1+1−1+... I can obtain 0 or 1. So the creation *ex nihilo* is quite plausible" (Bagni 1996, II) (we may note here the theological motivation for the argument, which may interest some pupils). Grandi's argument was based on bracketing the series in two alternative ways:

$$(1-1)+(1-1)+(1-1)+(1-1)+ \ldots = 0+0+0+\ldots = 0$$

$$1+(-1+1)+(-1+1)+(-1+1)+ \ldots = 1+0+0+0+\ldots = 1$$

The sum of the alternating series was considered ½ by many mathematicians in the 17th century. According to Grandi, this can be justified by considering the sum of the geometric series:

$$\frac{1}{1+x}=\sum_{i=0}^{+\infty}(-x)^i=1-x+x^2-x^3+\ldots$$

Then by putting $x = 1$ into the series we should have:

$$\sum_{i=0}^{+\infty}(-1)^i = 1-1+1-1+\ldots = ½$$

We may note that there is a simpler way to reach the same (false) conclusion. The same result can be achieved by the following procedure:

$$s = 1-1+1-1+\ldots \;\Rightarrow\; s = 1-(1-1+1-1+\ldots) \;\Rightarrow\; s = 1-s \;\Rightarrow\; s = ½$$

It can be interesting for some pupils to ask them what is the fallacy in this argument.

(It is because the argument works only once you have established that the series does indeed have a sum which is a number '*s*' like an y other; but that is what you are trying to establish. Nowadays we accept such a geometrical series as having a sum only if $|x|<1$).

Gottfried Wilhelm Leibniz, too, studied Grandi's series, and he wrote to Jacopo Riccati summarising the argument mentioned above:

> "I do not know if Mr. Count Riccati, and Mr. Zendrini have seen about the question whether $1-1+1-1$ etc. is ½, as R. P. Grandi stated, someway correctly. In fact $1/(1+x)$ is $1-x+xx-x^3+x^4$ etc. so if x is 1, we have $1/(1+1) = 1-1+1-1$ etc. = ½. It seems that this is clearly absurd. In the *Acta Eruditorum* from Leipzig I think I have solved this problem."

(this letter was probably written in 1715; see Michieli 1943, 579).

In fact Leibniz studied Grandi's series in some letters lo Christian Wolff, where he introduced an interesting probabilistic argument that influenced Johann and Daniel Bernoulli, too. Leibniz noticed that if we stop the series $1-1+1-1+\ldots$ at some finite stage, taken at random, it is possible lo have 0 or 1 with the same "probability". So the most "probable" value is the average between 0 and 1, so ½ (Leibniz 1715). This argument was accepted by some later distinguished mathematicians, notably Joseph Louis Lagrange and Siméon Denis Poisson.

Later in the 18th century, Leonhard Euler wrote in his textbook on differential calculus *Institutiones calculi differentialis* (1755):

> "We state that the sum of an infinite series is the finite expression by which the series is generated. From this point of view the sum of the infinite series $1-x+x^2-x^3+\ldots$ is $1/(1+x)$ because the series arises from the development of the fraction, for every value x".

Euler considered infinite series as a part of algebra of polynomials (Kline 1972, 537). So series were considered to be polynomials that can express the original function, without any convergence control. As we shall see, this situation can be important in the educational field.

Jacopo Riccati (Grugnetti 1985, 1986) criticised the convergence of Grandi's series io ½ in his *Saggio intorno al sistema dell'universo* (Riccati 1754/1761, 87), he wrote:

> "Grandi's argument is interesting, but it is wrong, because it causes contradictions. [...] Let us consider $n/(1+1)$ and, by the common procedure, lei us obtain the series $n-n+n-n+n-n$ et.cet. $= n/(1+1)$. If we remember that $1-1 = n-n$, or $1+n = n+1$, we have that either in this series or in Grandi's series there are the same number of 0".

The contradiction involving "the same number of 0" was reached in this way. Having written $½ = 1-1+1-1+\ldots$ "by the common procedure", Riccati introduced the series:

$$n/2 = n-n+n-n+n-n+\ldots$$

Let us compare the considered series, we can write:

$s = 1-1+1-1+1-1+ \ldots = (1-1)+(1-1)+(1-1)+ \ldots = 0+0+0+ \ldots$

$s' = n-n+n-n+n-n+ \ldots = (n-n)+(n-n)+(n-n)+ \ldots = 0+0+0+ \ldots$

So Riccati concluded that Grandi's procedure is incorrect. His argument cannot be accepted (notice that it is based upon the "common procedure", which is not correct for indeterminate series), although his conclusion is clear and correct (Riccati 1754/1761, 86):

"The mistake is caused by [...] the use of a series from which it is impossible to get any conclusion. In fact, [...] it does not happen that if we stop this series, the following terms can be neglected in comparison with preceding terms, and this property is verified only for convergent series".

Educational aspects

Let us now examine some educational aspects. This issue was raised with *Liceo Scientifico* students in Treviso (Italy) who did not know infinite series, although they had been introduced to the concept of infinite set. The following question was given to them (45 pupils 16-17 year olds and 43 17-18 year olds: 88 pupils in all):

"In 1703 the mathematician G. Grandi studied $1-1+1-1+1-1+ \ldots$ (addends, infinitely many, are always $+1$ and -1). What is your opinion about it?"

Pupils answered as follows:

26	pupils	(29%)	said the answer is 0
18	pupils	(20%)	said the answer "can be either 0 or 1"
5	pupils	(6%)	said the answer does not exist
4	pupils	(5%)	said the answer is ½
3	pupils	(4%)	said the answer is 1
2	pupils	(2%)	said the answer is infinite
30	pupils	(34%)	gave no answer.

First of all, notice that the greater part of the pupils interpreted the question as an implicit request to calculate the 'sum' of the series. Only 5 students (6%) explicitly stated that it is impossible to calculate the sum of Grandi's series. Note, too, that a fifth of the pupils suggested the possibility of two answers.

The students were interviewed about their answers. Some of them used, in effect, similar arguments to those found in the eighteenth century.

"If I want to add always 1 and -1, I can write $(1-1)+(1-1)$ so I can couple 1 and -1: I am going to add infinitely many 0, and I obtain 0:' (Marco, 3rd class, and 15 other pupils).

And those students who stated that the sum of the series is ½ justified it by arguments similar to the probabilistic argument of Leibniz:

"If I add the numbers I have 1, 0, 1, 0 and always 1 and 0. The average is ½." (Mirko, 4th class).

So students' justifications are remarkably similar to some we find in the history of mathematics. In particular, we can recognise, explicitly or implicitly, that some students felt as did several mathematicians in the 17th and 18th centuries, that an infinite series can be always considered a polynomial: the notion of convergence, not considered before the work of Gauss, had not entered the Italian pupils' heads yet either. This seems Io bear out in this instance the view of Piaget and Garcia (1983), that historical development and individual development are parallel.

Didactic reflection

Anna Sfard states that in order to speak of mathematical objects, it is necessary to make reference to the process of concept formation, and supposes that an operational conception can be considered before a structural one (Sfard 1991, 10). Concerning infinite series, the passage from an operational conception to a structural one is hard, because of the necessity of some basic notions (for example the limit concept).

As regards the *savoir savant*, the historical development of mathematical concepts can be considered as the sequence of (at least) two stages: an early, intuitive stage, and a mature stage; several centuries can pass between these stages. In the early stage the focus is mainly operational, the structural point of view is not a primary one. For example, in the early stage of working on infinite series (that is, at least until Gaussian works) main questions of convergence were not fully considered. From the educational point of view, a similar situation can be pointed out (Sfard 1991): of course, in an early stage pupils approach concepts by intuition, without a full comprehension of the matter. Then the learning becomes better and better, until it is mature.

There is a clear analogy between these situations. And the experimental results given above show that in the educational passage from the early stage to the mature one we can point out, in our pupils' minds, some doubts and some reactions that we can find in the passage from the early stage to the mature one as regards the *savoir savant*, too. Of course, processes of teaching and learning take place nowadays, after the full development of the *savoir savant*, with reference either to early stage, either to mature stage. So the didactic transposition, whose goal is initially a correct development of intuitive aspects, can be strongly based upon the results achieved in the mature stage, too, of the development of the *savoir savant*.

Moreover the process of teaching-learning and the didactic transposition must consider that, as we previously underlined, pupils' reactions are sometimes similar to corresponding reactions noticed in several great mathematicians in the history. This correspondence can be a very important tool for the teacher in developing the effectiveness of history as a resource base, but it needs a clear epistemological skill.

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