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Prime Numbers are Infinitely Many: Four Proofs from History to Mathematics Education

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Summary. The use of History into Mathematics Education links teaching-learning processes with historical elements. In this paper we discuss some epistemological issues related with the historical analysis of a mathematical topic, in order to achieve an effective and correct use of historical data into Mathematics Education. In particular we present some theoretical frameworks and underline the primary importance of the correct social and cultural contextualisation. Finally, we propose the comparison of some different strategies used by mathematicians in different historical periods in order to prove a theorem, with reference to presented theoretical frameworks.

1. History and Didactics: theoretical frameworks

The use of History into Mathematics Education links psychological learning processes with historical and epistemological issues (Radford, Boero & Vasco, 2000); it is an important topic of the research in Mathematics Education and the debate about it is open (Fauvel & van Maanen, 2000).

Concerning the interaction between History and Didactics, different levels can be considered: anecdotes presentation can be useful in order to strengthen pupils' conviction; higher levels bring out multidisciplinary relations and metacognitive possibilities (Furinghetti & Somaglia, 1997). These levels do not reflect just practical educational issues, but imply important epistemological assumptions (Radford, 1997): for instance, the selection of historical data is epistemologically relevant, and several problems are connected with their interpretation, always based upon our cultural institutions and beliefs (Gadamer, 1975).

From the historical point of view, frequently a new concept is encountered by Mathematicians in operative steps, like problem solving or proving activities; it will be theoretically framed many years or centuries later and finally it will assume the features that we (nowadays!) consider typical of real mathematical objects (Giusti, 1999). A similar evolution can be pointed out in the educational field: frequently the first contact with a new notion takes place in operative steps. A. Sfard notices that the development of "abstract mathematical objects" can be considered as the product of the comprehension of processes (Sfard, 1991; Slavit, 1997).

A parallelism between historical development and cognitive growth brings us to consider some epistemological problems: is it correct to present the History as a path that, by unavoidable mistakes, obstacles overcoming, critical reprises, leads to modern theories? What is the role played by social and cultural factors that influenced historical periods? It is necessary to overcome a merely evolutionary perspective: knowledge cannot be considered absolutely, according to a classical teleological vision; as we shall see, it must be understood in terms of cultural institutions (Radford, 1997).

Let us briefly present some theoretical frameworks.

- According to the "epistemological obstacles" perspective (Brousseau, 1983), a goal of historical study is finding systems of constraints (*situations fondamentales*) that must be studied in order to understand existing knowledge, whose discovery is connected to their solution (Radford, Boero & Vasco 2000, p. 163). Obstacles are clearly subdivided into epistemological, ontogenetic, didactic and cultural ones (Brousseau, 1989) and such subdivision points out that the sphere of the knowledge is considered isolately from other spheres. This perspective is characterised by other epistemological assumptions (Radford, 1997): the reappearance in teaching-learning processes, nowadays, of the same obstacles encountered by mathematicians in the past; and the exclusive, isolated approach of the pupil to the knowledge, without social interactions with other pupils and with the teacher (Brousseau, 1983).

Epistemological assumptions needed by the mentioned perspective are relevant. Let us underline that it is impossible, nowadays, to see historical events without the influence of our modern conceptions (Gadamer, 1975); so we are forced to consider the following dilemma: should we resign historical references and their educational uses, in order to avoid their pollution caused by our conceptions of the past? Otherwise we must accept our modern point of view and take into account that, when we look at the past, we connect two cultures that are "different [but] they are not incommensurable" (Radford, Boero & Vasco, 2000, p. 165; Furinghetti & Radford, 2002).

- According to the socio-cultural perspective by L. Radford, knowledge is linked to activities of individuals and, as we noticed, this is strictly related to cultural institutions (Radford, 1997); knowledge is not built individually, but into a wider social context (Radford, Boero & Vasco, 2000, p. 164). The role played by History must be interpreted with reference to different socio-cultural situations and moreover it gives us the opportunity for a deep critical study of considered historical periods. Another important approach is the "voices and echoes" perspective by P. Boero (Boero & Al. 1997 and 1998).

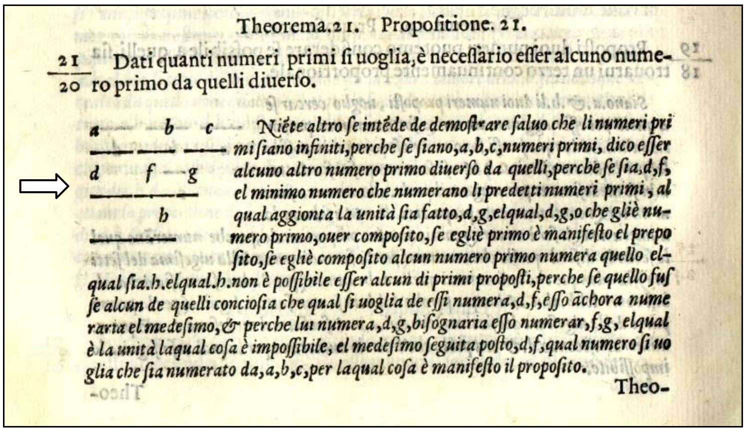
2. Prime numbers are infinitely many

In our opinion, the comparison of some different strategies used by mathematicians in different historical periods in order to prove a theorem can be interesting with reference to theoretical frameworks previously sketched. We shall consider the Proposition IX-20 of Euclid's *Elements* which states that prime numbers are infinitely many (the original statement is in potential sense: Ribenboim, 1980, p. 3):

Prime numbers are more than any assigned multitude of prime numbers.

We consider four proofs of this celebrated theorem (there are a lot of different possibilities, so our choice is epistemologically relevant, see for instance: Ribenboim, 1980 and Aigner & Ziegler, 1998), by Euclid, Euler, Erdös and Fürstenberg:

I. ***Euclid*: 300 b.C.** (N. Tartaglia, 1569, p. 171; F. Commandino, 1619, p. 118). Let $p_1 = 2 < p_2 = 3 < \ldots < p_r$ be primes and $Q = p_1 \cdot p_2 \cdot \ldots \cdot p_r + 1$; p is a prime that divides Q; then p cannot be one of the $p_1, p_2, \ldots, p_r$, because p cannot divide the difference $Q - p_1 \cdot p_2 \cdot \ldots \cdot p_r = 1$. So $p_1, p_2, \ldots, p_r$ are not all the prime numbers. ■


Theorema.21. Propositione. 21.

21/20 Dati quanti numeri primi si uoglia, è necessario esser alcuno numero primo da quelli diuerso.

Niẽte altro se intẽde de demostrare saluo che li numeri primi siano infiniti, perche se siano, a, b, c, numeri primi, dico esser alcuno altro numero primo diuerso da quelli, perche se sia, d, f, el minimo numero che numerano li predetti numeri primi, al qual aggionta la unità sia fatto, d, g, elqual, d, g, o che gliè numero primo, ouer composito, se egliè primo è manifesto el preposito, se egliè composito alcun numero primo numera quello elqual sia. h. elqual. h. non è possibile esser alcun di primi proposti, perche se quello fusse alcun de quelli conciosia che qual si uoglia de essi numera, d, f, esso ãchora numeraria el medesimo, & perche lui numera, d, g, bisognaria esso numerar, f, g, elqual è la unità laqual cosa è impossibile, el medesimo seguita posto, d, f, qual numero si uoglia che sia numerato da, a, b, c, per laqual cosa è manifesto il proposito.

Theo-

In this edition of Euclid's *Elements* (Tartaglia, 1569, p. 171), the visual representation of numbers can be referred to Greek *Geometric Algebra*; the proof is expressed in a verbal register (the register available at the time, of course). It is necessary to take into account either the period in which the original work was written (300 b.C.), either the period of its edition (XVI century: Barbin, 1994)

I. ***Leonhard Euler*: 1737 and 1748** (Ribenboim, 1980, pp. 7-8 and 155-157).

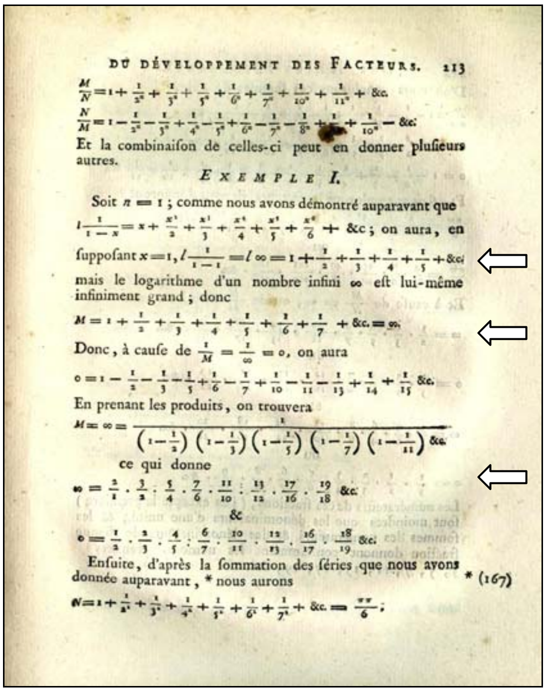
Let us consider, being $|x| < 1$: $\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$. By putting $x = \frac{1}{2}$, $x = \frac{1}{3}$ we have:

$$\frac{1}{1-\frac{1}{2}} = \sum_{\alpha=0}^{+\infty} \frac{1}{2^\alpha} \text{ and } \frac{1}{1-\frac{1}{3}} = \sum_{\beta=0}^{+\infty} \frac{1}{3^\beta} \text{ so: } \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = \sum_{\alpha,\beta}^{0,+\infty} \frac{1}{2^\alpha}\frac{1}{3^\beta}$$

On the right we have 1 ($\alpha = \beta = 0$) and the inverses of positive integers having only prime factors 2, 3. If prime numbers were finitely many, $p_1, p_2, \ldots, p_m$:

$$\frac{1}{1-\frac{1}{p_1}} \cdot \frac{1}{1-\frac{1}{p_2}} \cdot \ldots \cdot \frac{1}{1-\frac{1}{p_n}} = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n}^{0,+\infty} \frac{1}{p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}}$$

where on the right we have the *harmonic series*. But the quantity on the left would be finite and the harmonic series diverges: this is absurd. ■



DU DÉVELOPPEMENT DES FACTEURS. 213

$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$

$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.$

Et la combinaison de celles-ci peut en donner plusieurs autres.

EXEMPLE I.

Soit $n = 1$; comme nous avons démontré auparavant que $l\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c$; on aura, en supposant $x = 1$, $l\frac{1}{1-1} = l\,\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$ mais le logarithme d'un nombre infini ∞ est lui-même infiniment grand ; donc

$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = \infty.$

Donc, à cause de $\frac{1}{M} = \frac{1}{\infty} = 0$, on aura

$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15}\ \&c.$

En prenant les produits, on trouvera

$M = \infty = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\ \&c.}$

ce qui donne

$\infty = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18}\ \&c.$

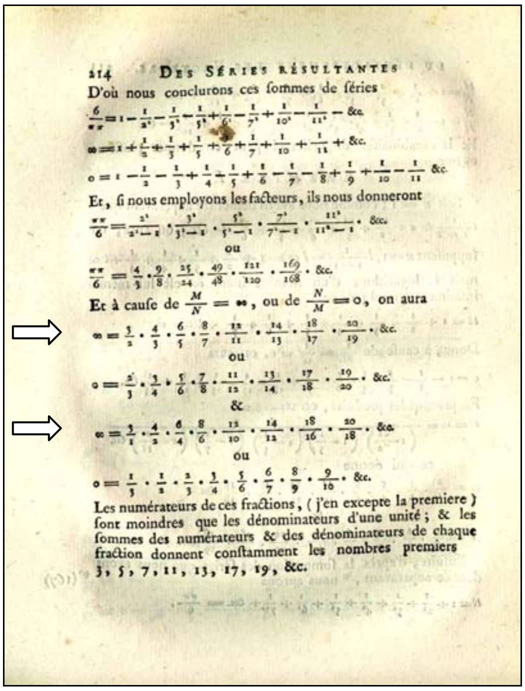
&

$0 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19}\ \&c.$

Ensuite, d'après la sommation des séries que nous avons donnée auparavant, * nous aurons

* (167)

$N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6};$



214 DES SÉRIES RÉSULTANTES

D'où nous conclurons ces sommes de séries

$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$

$\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \&c.$

$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11}\ \&c.$

Et, si nous employons les facteurs, ils nous donneront

$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \&c.$

ou

$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} \cdot \frac{49}{48} \cdot \frac{121}{120} \cdot \frac{169}{168} \cdot \&c.$

Et à cause de $\frac{M}{N} = \infty$, ou de $\frac{N}{M} = 0$, on aura

$\infty = \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \&c.$

ou

$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \&c.$

&

$\infty = \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{6}{4} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{20}{18} \cdot \&c.$

ou

$0 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \&c.$

Les numérateurs de ces fractions, (j'en excepte la premiere) sont moindres que les dénominateurs d'une unité ; & les sommes des numérateurs & des dénominateurs de chaque fraction donnent constamment les nombres premiers 3, 5, 7, 11, 13, 17, 19, &c.

Some Euler's notations and procedures (Euler, 1796, I, pp. 213-214) would not be considered "rigorous" according to our modern standards; but formal correctness must be always investigated in its own conceptual context and not against contemporary standards, in order to avoid the imposition of modern conceptual frameworks to works based upon different ones

Moreover Euler gave the following proof that the series $\Sigma 1/p$, being p primes, diverges (see: Tenenbaum & Mendès France, 1997, pp. 23-24): every positive integer n can be written in a unique way as the product of a square-free number q and of m^2; let q be a square-free number; we have:

$$\sum_{n\le x}\frac{1}{n}=\sum_{q\le x}\left(\frac{1}{q}\sum_{m\le\sqrt{x/q}}\frac{1}{m^2}\right)\le\sum_{q\le x}\left(\frac{1}{q}\sum_{m=1}^{+\infty}\frac{1}{m^2}\right)$$

and: $$\sum_{m=1}^{+\infty}\frac{1}{m^2}\le 1+\sum_{m=2}^{+\infty}\frac{1}{(m-1)m}=1+\sum_{m=2}^{+\infty}\left(\frac{1}{m-1}-\frac{1}{m}\right)=2$$

so: $$\sum_{n\le x}\frac{1}{n}\le 2\sum_{q\le x}\frac{1}{q}$$

Let us now consider $\sum_{q\le x}\frac{1}{q}$, being p a prime number:

$$\sum_{q\le x}\frac{1}{q}\le\prod_{p\le x}\left(1+\frac{1}{p}\right)\le\exp\left\{\sum_{p\le x}\frac{1}{p}\right\}$$

(by developing $\prod_{p\le x}\left(1+\frac{1}{p}\right)$ and from: $1+a\le e^a$ being $a=1/p$). So:

$$\sum_{n\le x}\frac{1}{n}\le 2\exp\left\{\sum_{p\le x}\frac{1}{p}\right\}.$$

From $\frac{1}{n}\ge\int_n^{n+1}\frac{dt}{t}$, we have: $\sum_{n\le x}\frac{1}{n}\ge\sum_{n\le x}\int_n^{n+1}\frac{dt}{t}\ge\log x$ and:

$$\log x\le\sum_{n\le x}\frac{1}{n}\le 2\exp\left\{\sum_{p\le x}\frac{1}{p}\right\},\text{ so: }\sum_{p\le x}\frac{1}{p}\ge\log\log x-\log 2.$$

Being $\lim_{x\to+\infty}\log x=+\infty$, we conclude that $\Sigma 1/p$ diverges. ■

II. ***Paul Erdös*: 1938** (Erdös, 1938; Aigner & Ziegler, 1998, p. 6).

Erdös, too, proved that the series $\Sigma 1/p$, being p prime numbers, diverges. Let $p_1 = 2 < p_2 = 3 < p_3 < \ldots$ be the primes (in increasing order). If the series $\Sigma 1/p$ would converge, then there would be a positive integer k such that: $\sum_{i\ge k+1}\frac{1}{p_i}<\frac{1}{2}$.

Let us call $p_1, \ldots, p_k$ *small primes* and p_{k+1}, p_{k+2} *great primes*. Let N be any positive integer; we can write: $\sum_{i\ge k+1}\frac{N}{p_i}<\frac{N}{2}$.

Let N_b be the number of the positive integers $n\le N$ divisible for (at least) a great prime and let N_s be the number of the positive integers $n\le N$ divisible only for small primes. We shall prove that there is N such that $N_b+N_s < N$ and this is absurd (in fact: $N_b+N_s = N$).

$\left\lfloor \frac{N}{p_i} \right\rfloor$ is the number of positive integers $n \leq N$ that are multiple of p_i. So from

$\sum_{i \geq k+1} \frac{N}{p_i} < \frac{N}{2}$ it follows: $N_b \leq \sum_{i \geq k+1} \left\lfloor \frac{N}{p_i} \right\rfloor < \frac{N}{2}$.

Concerning N_s, we have previously underlined that every positive integer n can be written in a unique way as the product of a square-free number q and of m^2; let us write every $n \leq N$ having only small prime divisors as $n = a_n b_n^2$, being a_n square-free. So every a_n is a product of different small primes and there are exactly 2^k different square-free parts. Moreover, being $b_n \leq \sqrt{n} \leq \sqrt{N}$, there are at most $\sqrt{N}$ square parts, so: $N_s \leq 2^k \sqrt{N}$.

Being $N_b \leq \sum_{i \geq k+1} \left\lfloor \frac{N}{p_i} \right\rfloor < \frac{N}{2}$ true for every N, in order to achieve the *reductio ad absurdum* we have to find a number N such that $2^k \sqrt{N} \leq \frac{N}{2}$ i.e. $2^{k+1} \sqrt{N} \leq N$; it is $N = 2^{2k+2}$ so with reference to this number we would have $N_b+N_s < N$. ■

III. ***Harry Fürstenberg*: 1955** (Fürstenberg, 1955; Ribenboim, 1980, pp. 11; Aigner & Ziegler, 1998, p. 5; Fürstenberg's ideas are reprised in: Golomb, 1959).

Let **Z** be the set of integers, $a \in$ A, $b \in$ B and: $N_{a,b} = \{a + nb : n \in \mathbf{Z}\}$.

We shall call the set A *open* if A is ∅ or if for every $a \in$ A there is a positive integer b such that $N_{a,b}$ is a subset of A; it is well known that every union of open sets is open; if A_1, A_2 are open, $a \in A_1 \cap A_2$ being $N_{a,b_1} \subseteq A_1, N_{a,b_2} \subseteq A_2$, so: $a \in N_{a,b_1 b_2} \subseteq A_1 \cap A_2$: so every finite intersection of open sets is open.

It follows that the described family of open sets induces a topology in **Z**.

Let us notice that every non-empty open set is infinite. Moreover every $N_{a,b}$ is closed, being: $N_{a,b} = \mathbf{Z} \setminus \bigcup_{i=1}^{b-1} N_{a+i,b}$ so $N_{a,b}$ is the complement of an open set.

Every integer different from 1 and from –1 has at least a prime divisor p so it belongs to $N_{0,p}$; so: $\mathbf{Z} \setminus \{1, -1\} = \bigcup_{p \in \mathbf{P}} N_{0,p}$.

If the set of prime number were finite, then $\bigcup_{p \in \mathbf{P}} N_{0,p}$ would be a finite union of closed sets, so closed; hence $\{1, -1\}$ would be an open set: this is absurd. ■

3. A comparison of the quoted proofs

First of all, let us notice that all such arguments are *proofs*, with the usual meaning nowadays ascribed to such word. Euclidean *Elements*, for instance, are placed after the passage from the empirical Greek Mathematics to the deductive Mathematics, in a socio-cultural context based upon the distinction between real knowledge and opinions (drawn by Parmenides: Szabó, 1977) and a social intellectual habit consisting of a particular style of argumentation (Radford, 1996 and 1997). Moreover it is interesting to underline the use of the *reductio ad absurdum*: concerning Euclid's proof, this element can be related with the "Being/non-Being" ontological structure of the considered period (Radford, 2003, p. 70).

We shall present some features of the quoted proofs.

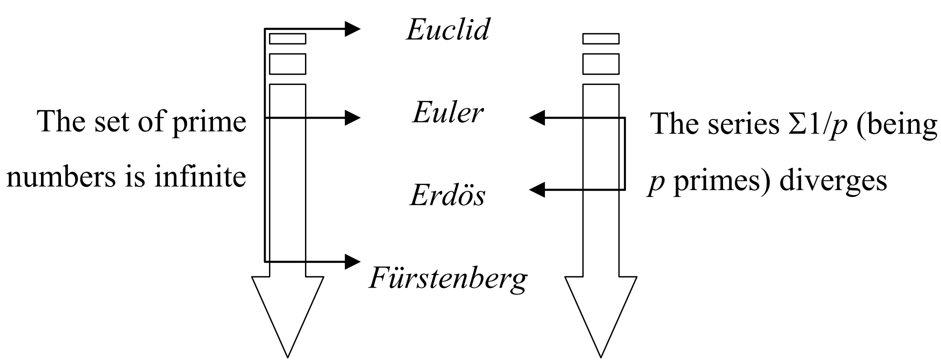
Author and date	**Proved statement**	**Logical structure**	**Conception of infinity**	**Mathematical context**
Euclid (300 b.C.)	Prime numbers are more than any assigned multitude of prime numbers	*Reductio ad absurdum*	Exclusively potential infinity	Basic Arithmetics
Euler (1737 and 1748)	Prime numbers are infinitely many. The series Σ1/p (being p primes) diverges	*Reductio ad absurdum*	He considered the infinite series $\sum_{i=0}^{\infty}\frac{1}{p^i}=\frac{1}{1-(1/p)}$ With reference to series, infinity is potential	Use of some analytical notions
Erdös (1938)	The series Σ1/p (being p primes) diverges	*Reductio ad absurdum*	Explicit use of infinite series	Basic Number Theory
Fürstenberg (1955)	The set of prime numbers is infinite	*Reductio ad absurdum*	Actual infinity	Basic Topology

Clearly considered proofs are developed in different mathematical sectors; but the crucial point is that they were proposed in very different historical and socio-cultural contexts. So the main question is the following: is it correct to consider the quoted proofs as *four different proofs of the same theorem*? In our opinion the answer is: *no*. The celebrated proposition according to which prime numbers are infinitely many is just the hint, the early idea that stimulated different mathematicians, in different periods, to develop different important mathematical contents.

Nowadays, let us examine some educational possibilities connected to the presentation of the quoted proofs. We have underlined that it would not be meaningful to state that they make reference to a similar epistemological obstacle; they are not referred to four *situations fondamentales*: when Euler or Fürstenberg proved the infinity of prime numbers, they knew ancient Euclid's result and approached the problem according their own conceptions. So these proofs allow us to compare the different cultural contexts of the periods in which they were conceived, with reference to different cultural institutions and beliefs, and this is the fundamental issue.

For instance, let us present some remarks:

- First of all, proved statements are remarkably different: Euclid considers "a given quantity of prime numbers" (nowadays we should say: "a set of prime numbers"). Euler and Erdös prove that the infinite series $\Sigma 1/p$, being p primes, diverges; and this is sufficient (but not necessary) in order to state that prime numbers are infinitely many: it is interesting to underline that in XVIII century the focus is mainly operational. In the proof by Fürstenberg, the reference to the set or prime numbers is explicit. So we can consider two different approaches:



Differences: conception of infinity; mathematical contexts; representation registers employed

Differences: mathematical contexts; "rigour" in using infinite series

- The particular conception of infinity is a crucial element in order to comprehend the sense of the mentioned proofs: Euclid considers potential infinity, following Aristotle (*Physics*, Γ, 6-7, 207a, 22-32) and according to the cultural institutions and the beliefs of his own time; Euler and Erdös make reference to a series, so to a process, and Fürstenberg considers an infinite set in actual sense.
- The connection between Mathematics and socio-cultural context is fundamental: for instance, Euler's approach by infinite series is not just "tuned in" to applicative features of the scientific frame of mind in the XVIII century (Crombie, 1995). The influence of non-mathematical elements is complex and deep.
- Apart from different mathematical contexts, we noticed that a difference between Euler and Erdös regards the "rigour". But what do we mean by that? Formal correctness must be investigated in its own conceptual context and not against contemporary standards, in order to avoid the imposition of modern conceptual frameworks to works based upon different ones: so Euclid and Euler *were* rigorous *in their own ways*. This remark imply some issues related to the educational use of original sources: when we consider Euler's proofs nowadays, for instance in classroom practice, we often *rewrite* them according to our standards: so, by that, really we are looking at the past through our "non-transparent lens" (Confrey & Smith, 1994, p. 173). As noticed, probably this is unavoidable: but we must always keep it in mind.
- Representation registers are influenced by considered historical periods: however, concerning Euclid's proof, it is important to take into account either the period in which the original argument was conceived (300 b.C.), either the period of the considered editions (Tartaglia, 1569; Commandino, 1619). In Euclid, "placed within the Eleatan-Platonic mode of knowing" (Radford, 2003), we don't find visual methods used, for instance, in the sense of Pythagoreans (concerning the Greek *Geometric Algebra*, see: Kline, 1972); of course the status of visualisation in XVI century is different (we suggest to consider: Bombelli, 1572, in particular the 3rd Book) and it influences the quoted editions of *Elements*. Euler makes reference to diagrams and integrals in his proof of the divergence of $\Sigma 1/p$; later, the importance of symbolic registers seems to be progressively increasing. Of course a complete study would consider the various

particular registers used, for instance, by Euler, by Erdös or by Fürstenberg; in fact there is not a single register of a given kind: the nature of a register depends on the community of practice in question.

4. Final reflections

A wider research will provide a detailed analysis of mentioned proofs with reference to their respective socio-cultural contexts and to their comparison (concerning the primary role played by semiotic aspects see: Radford, 2003, where *Cultural Semiotic Systems* are presented). We now propose some reflections:

a. Euclid's proof must be considered in relation to Greek intellectual habits.
a. Euler's approach must be considered in relation to socio-cultural situation of XVIII century.
c. The comparison between Euler and Erdös allows us to underline that rigour must be evaluated in its own conceptual context.
d. Different notions of infinity, for instance in Euclid and in Fürstenberg, are related to different social and philosophical contexts.
e. Different representation registers must be considered with reference to communities of practice in question, either in the period in which the original works were written, either in the period of their editions.

More generally, in the first paragraph we stated that History of Mathematics gives us important educational opportunities:

- the possibility of a metacognitive reflection;
- the possibility to achieve a wide comprehension of historical periods.

These possibilities are indivisibly linked: in fact the transfer of some situations from History to Didactics cannot be state just by analogy, but it needs a wider cultural dimension that must keep into account non-mathematical elements, too (Radford, 1997). Clearly the presented perspective would require a good epistemological skill of teachers and pupils. However, in our opinion, an "internalist" History, so a conception of the development of Mathematics as a pure subject, isolated from non-mathematical "external" influences, is hardly useful in Mathematics Education (Grugnetti & Rogers, 2000, p. 40; Bagni, forthcoming) and brings to relevant epistemological problems. From this point of view, with reference to aforementioned educational opportunities, the former can be justified by the latter.

Original books

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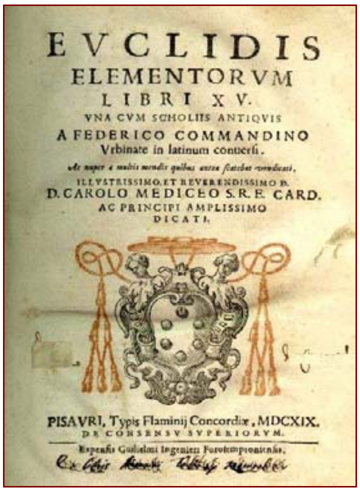
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ELEMENTORVM
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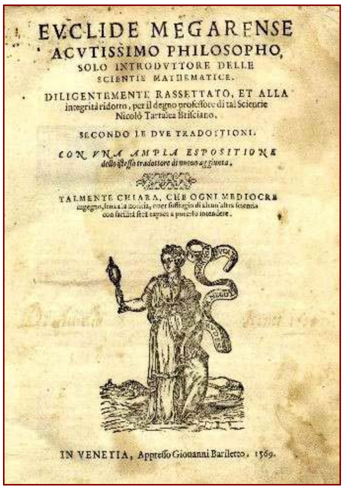
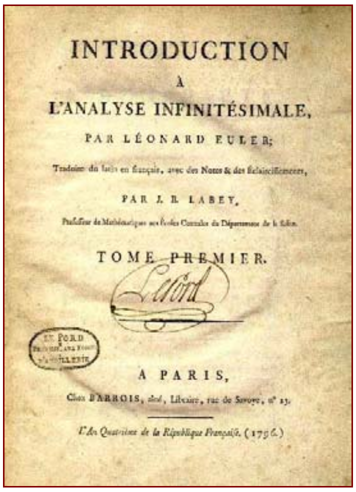

EVCLIDE MEGARENSE
ACVTISSIMO PHILOSOPHO,
SOLO INTRODVTTORE DELLE
SCIENTIE MATHEMATICE.
DILIGENTEMENTE RASSETTATO, ET ALLA
integrità ridotto, per il degno professore di tal Scientie
Nicolò Tartalea Brisciano.
SECONDO LE DVE TRADOTTIONI.
TALMENTE CHIARA, CHE OGNI MEDIOCRE
IN VENETIA, Appresso Gioanni Bariletto. 1569.


INTRODUCTION
À
L'ANALYSE INFINITÉSIMALE,
PAR LÉONARD EULER;
PAR J. B. LABEY,
TOME PREMIER.
A PARIS,
L'An Quatrième de la République Française. (1796.)

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Concerning the present paper, we particularly underline the importance of L. Radford's paper (Radford, 1997):

http://laurentian.ca/educ/lradford/FLM%2097%20(final%20version).htm
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