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Didactics and history of numerical series, 100 years after Ernesto Cesaro's death (1906): Guido Grandi, Gottfried Wilhelm Leibniz and Jacopo Riccati

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Summary. *In this paper, an historical example about numerical series is presented and its educational utility is investigated and discussed. Students' behaviour is examined by a test and by some interviews (students aging 16-18 years): we conclude that the problem of the passage from finite to infinite is mainly a cultural one, and historical issues are important in order to approach it, although the historical perspective can be considered together with other relevant approaches and viewpoints, e.g. R. Rorty's "irony".*
Keywords: convergence, didactical contract, history of mathematics, infinite series, probabilistic argument, summability.

Historical introduction: Guido Grandi's series

According to Hans-Georg Gadamer (2000, pp. 809-811), «*to think historically* actually means *to carry out completely the transposition that concepts of the past go through* when we try to think on the basis of them. [This] always implies a mediation between [historical] concepts and our thinking» (in this paper the translations are ours); and Paul Karl Feyerabend (1996, p. 17) notices that «our vision of the history is

influenced by some models that hypnotize us» (see moreover: D'Amore, Radford, Bagni, 2006). In the present paper we shall apply these ideas in the discussion of some educational possibilities related to an historical example, a celebrated indeterminate series.
In 1703, Guido Grandi noticed that from 1−1+1−1+... it is possible to obtain both the "results" 0 and 1:

(1−1)+(1−1)+(1−1)+... = 0 1+(−1+1)+(−1+1)+... = 1

The "sum" of this numerical series was considered ½ by Grandi. According to him, a proof of this statement can be based upon the following expansion (expressed by using modern notation), nowadays accepted if and only if $|x| < 1$:

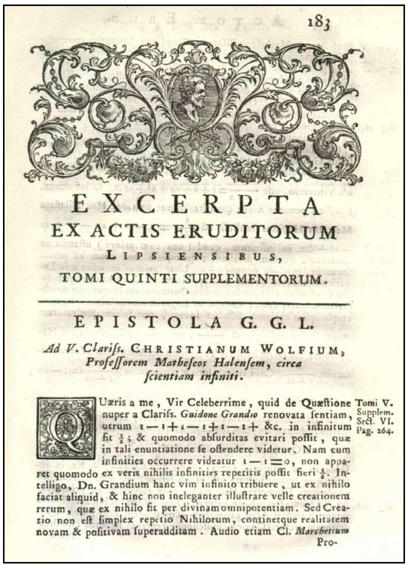
$$\frac{1}{1+x} = \sum_{i=0}^{+\infty}(-x)^i = 1 - x + x^2 - x^3 + \dots$$

From $x = 1$ (and of course this is *not* correct) we should have the equality: 1–1+1–1+... = ½.
However we have to take into account the following issue: *did the term "convergence" (with its modern meaning) belong to Guido Grandi's vocabulary?* So could we propose a correct historical analysis of Grandi's series on the basis of the notion of convergence? And what about educational implications?

Series in the 18th century: Leibniz and Riccati

Gottfried Wilhelm Leibniz (1646-1716) studied Guido Grandi's series in some letters (1713-1716) to German philosopher Christian Wolff (1678-1754), where Leibniz introduced the "probabilistic argument" (that influenced, for instance, Johann and Daniel Bernoulli).
Leibniz noticed that if we "stop" the infinite series 1–1+1–1+... (see: Leibniz, 1716, 187), it is possible to obtain either 0 or 1 with the same "probability". As a matter of fact, «the *series finita* […] can have an even number of terms, and the final one is negative: 1–1, or 1–1+1–1, or 1–1+1–1+1–1 […] or it can have an odd number of terms, and the final one is positive: 1, or 1–1+1, or 1–1+1–1+1 and so on» (Leibniz, 1716, p. 187).

So Leibnitian conclusion is the following: «when numbers' nature vanishes, our possibility to consider even or odd numbers vanishes, too. […] So taking into account what is stated by the authors that wrote about evaluations, […] we ought to take the arithmetic mean [of 0 and 1], i.e. the half of their sum; and in this case nature itself respects *justitiae* law» (Leibniz, 1716, 187). Hence the "most probable" value to be chosen is the arithmetic mean of 0 and 1, that is ½.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EXCERPTA
EX ACTIS ERUDITORUM
LIPSIENSIBUS,
TOMI QUINTI SUPPLEMENTORUM.

EPISTOLA G. G. L.

Ad V. Clarifs. CHRISTIANUM WOLFIUM, *Profeſſorem Matheſeos Halenſem, circa ſcientiam infiniti.*

QUæris a me, Vir Celeberrime, quid de Quæſtione nuper a Clariſs. *Guidone Grandio* renovata ſentiam, utrum 1 — 1 + 1 — 1 + 1 — 1 + &c. in infinitum fit ½; & quomodo abſurditas evitari poſſit, quæ in tali enuntiatione ſe oſtendere videtur. Nam cum infinities occurrere videatur 1 — 1 = 0, non apparet quomodo ex veris nihilis infinities repetitis poſſit fieri ½. Intelligo, Dn. Grandium hanc vim infinito tribuere, ut ex nihilo faciat aliquid, & hinc non ineleganter illuſtrare velle creationem rerum, quæ ex nihilo fit per divinam omnipotentiam. Sed Creatio non eſt ſimplex repetio Nihilorum, continetque realitatem novam & poſitivam ſuperadditam. Audio etiam Cl. *Marchettum* Pro- Tomi V. Supplem. Sect. VI. Pag. 264.

Forty years later, Jacopo Riccati (1676-1754) criticised the convergence of Guido Grandi's series to ½; in his last treatise, *Saggio intorno al sistema dell'universo* (published in 1754), Riccati wrote: «[Grandi's] argument is interesting, but wrong because it causes contradictions. […] The mistake is caused by the use of a series […] from which it is impossible to get any conclusion. In fact, […] it does not happen that the following terms can be neglected in comparison with preceding terms; this property is verified only for convergent series» (Riccati, 1761, I, p. 87). In fact, Riccati made reference to some fundamental keywords referred to convergence: as a consequence, his vocabulary is essentially different from Grandi's one.

History and mathematics education

Let us now briefly consider our students' opinions regarding Grandi's series. A test (whose results are discussed in: Bagni, 2005) has been proposed to students of two third-year *Liceo Scientifico* classes, total 45 students (aging 16-17 years), and of two fourth-year *Liceo scientifico* class, 43 students (aging 17-18 years; total: 88 students), in Treviso. Italy. Their mathematical curricula were traditional: in all classes, at the moment of the test, students did not know infinite series. We asked our students to consider the expression "1–1+1–1+..." (studied "in 1703" by "the mathematician Guido Grandi"), taking into account that "addends, infinitely many, are always +1 and –1" and to express their "opinion about it" (time: 10 minutes; no books or calculators allowed). Some students stated that the sum of the considered series is ½ and they made reference to justifications similar to Leibnitian "probabilistic argument". Audio-recorded material allowed us to point out a salient short passage (1 minute and 35 seconds, 9 utterances):

[1] Researcher: "Why did you write that the result is ½?"
[2] Mirko: "Oh, well, I start with 1, so I have 0, then 1, 0 and so on. There are infinitely many +1 and –1."
[3] Researcher: "That's true, but how can you say ½?"
[4] Mirko: "If I add the numbers, I obtain 1, 0, 1, 0 and always 1 and 0. The mean is ½."
[5] Researcher: "And so?"
[6] Mirko: "The numbers that I find are 1, 0, and 1, 0, and 1, 0 and so on: clearly, for every couple of numbers, one of them is 0 and one of them is 1. So these possibilities are equivalent and their mean is ½."
[7] Mirko: [*after 12 seconds*] "Perhaps my answer is strange, or wrong, but I don't see a different correct result: surely both the results 0 and 1 are wrong. If I say that the result is one of that numbers, for instance 1, I forget all the other numbers, an infinite sequence of 0."
[8] Researcher: "So in your opinion both 0 and 1 cannot be considered the correct answer."
[9] Mirko: "Alright, and in this case what is the result? I wrote that ½ is the result of the operation because ½ is the mean, so it is a number that, in a certain sense, contains both 0 and 1."

Mirko stated that "for every couple of numbers, one of them is 0 and one of them is 1" ([4]) and "the mean […] is a number that, in a certain sense, contains both 0 and 1"([9]). So Mirko made reference to algebraic procedures: a series, in his opinion, is a kind of algebraic operation; it is necessary "to add" the numbers in order to obtain the (one and only) "result". Let us remember that Grandi's series has been expressed in the form "1–1+1–1+...", with the remark "addends, infinitely many, are always +1 and –1": the language is typically algebraic, and several students made reference to "algebraic rules".
So Mirko did not make explicit reference to the probability: he mainly tried to find a result for the considered problem, and clearly this is an

educational issue (influenced by the didactical contract); in the 18th century, the probabilistic argument was based upon a slightly different remark, according to which if we "stop" the infinite series 1−1+1−1+..., it is possible to obtain both the "sums" 0 and 1 with the same "probability".
Apart from this difference, what is, nowadays, the correct reaction to be assumed by the teacher? Of course, to state "Grandi's series converges" is wrong; but our reaction, as we shall see, would require irony (in the sense of: Rorty, 2003, pp. 89-90).

History and mathematics education

We know that Grandi's series is indeterminate; nevertheless… it "converges", for instance, in the sense of Georg Frobenius (1849-1917); this notion is based upon ideas of Daniel Bernoulli and Joseph Raabe (1801-1859), and has been generalized by Ludwig Otto Hölder (1859-1937) and by Ernesto Cesàro (1859-1906). Given the numerical series $a_0+a_1+a_2+\ldots$, let us consider the sequence:

$$s_0 = a_0; \quad s_1 = a_0+a_1; \quad s_2 = a_0+a_1+a_2 \quad \text{etc.}$$

For every $n\geq 0$ let σ_n be the arithmetic mean of $s_0, s_1, \ldots, s_n$. We say that a series converges "according Frobenius-Cesàro" if the sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ converges (usually). With reference to Grandi's series, sequence $\sigma_0, \sigma_1, \ldots$ is:

$$1,\ 1/2,\ 2/3,\ 1/2,\ 3/5,\ 1/2,\ 4/7,\ \ldots$$

and it *converges* (usually) to ½. So the statement "Grandi's series does not converge" could be criticised: it requires «irony», in Richard Rorty's words, i.e. a subject's frame of mind to discuss his/her own vocabulary, and the awareness that this vocabulary is not «closer to reality than others» (Rorty, 2003, p. 89).
Let us remember that in the 20th century many mathematical techniques based upon the "convergence" (summability) according to Frobenius-Cesàro have been applied, for instance, to Fourier series. So our example suggests a wider educational approach, in order to consider different experiences that give sense to our mathematical language.
The main problem of the passage from finite to infinite is a cultural one, and historical issues are important in order to approach it, although, undoubtedly, the historical approach is to be considered together with

other educational approaches (see: Radford, 1997): this problem requires an adequate mediation (Gadamer, 2000, p. 811) and it can take into account Rortyan «irony».

In our opinion, previous examples show us that an "ironic mathematics" can be "mathematically" very deep. And they can induce our students to "open their eyes", to have a look, awarely, at mathematics itself, and at the world, without the problems connected to technicality; without, in Martin Heidegger's words, the oppressive influence of «calculating thought» (Heidegger, 1982).

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