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"HISTORY OF CALCULUS FROM EUDOXUS TO CAUCHY"
Historical investigation and interpretation and Mathematics education

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ABSTRACT

A. In this paper we consider some issues concerning the historical presentation of a mathematical concept, with particular reference to the educational introduction of infinitesimal methods: we present some theoretical frameworks and highlight the importance of non-mathematical elements.

B. We propose an historical survey, taking into account some approaches by Euclid, Cavalieri, Wallis, Leibniz, Euler, d'Alembert, Lagrange and Cauchy.

C. We conclude that educational introduction of infinitesimal methods by historical references requires a socio-cultural contextualization; nevertheless, historiographically, an aprioristic platonic epistemological perspective is frequently assumed.

1. Theoretical preface

In his *The Essential Tension*, T.S. Kuhn offers students the maxim: “When reading the works of an important thinker, look first for the apparent absurdities in the text and ask yourself how a sensible person could have written them. When you find an answer (…) then you may find that more central passages, ones you previously thought you understood, have changed their meaning” (Kuhn, 1977, p. xii). This quotation will provide us with an important hint in order to evaluate some historical contributions and to use them correctly and effectively in educational practice.

Some theoretical frameworks can be mentioned in order to link learning processes with historical issues. According to the epistemological obstacles perspective, some systems of constraints in the History must be studied for understanding existing knowledge. Obstacles are subdivided into epistemological, ontogenetic, didactic and cultural (Brousseau, 1989), so knowledge is separated from the other spheres; an important assumption is connected to the reappearance in teaching-learning processes of the same obstacles encountered by mathematicians in the History; the isolated approach of a pupil to the knowledge, without social interactions with other pupils and with the teacher, is moreover remarkable.

However can we directly compare different historical periods? What is the role played by socio-cultural factors? It is impossible, nowadays, to interpret historical events without the influence of modern conceptions (Furinghetti, Radford, 2002); so we must accept our point of view and take into account that, when we look at the past, we connect two cultures that are “different [but] they are not incommensurable” (Radford, Boero, Vasco, 2000, p. 165). According to Radford’s socio-cultural perspective, knowledge is linked to activities of individuals and related to cultural institutions; it is built into a social context and the educational role played by historical elements must be considered with reference to different socio-cultural situations (Radford, 1997 and 2003).

In our opinion, knowledge can hardly be considered according to a classical teleological vision: let us explain this by a first example. In his *Quadratura parabolae*, Archimedes (287-212 BC) proved the following Proposition 23 (see: Euclid’s *Elements*, IX-35): “If some quantities are such that everyone of them is four times the following one, all these quantities plus the third part of the lowest are 4/3 of the greatest one” (Frajese, 1974, pp. 511-512. In this paper the translations are ours).

Many centuries later, F. Viète (1540-1603) calculated the sum of a geometric infinite series; and in 1655 A. Tacquet (1612-1660) published a similar result (see moreover Wallis’ *Arithmetica infinitorum*) and stated: “It is amazing that [ancient] mathematicians, who knew the theorem concerning finite progressions, did not consider the result concerning infinite ones, that can be immediately deduced by such theorem” (in: Loria, 1929-1933, p. 517).

Tacquet made reference to ancient mathematics with no historical contextualization; although Aristotle implicitly noticed that the sum of a great number of addends (an infinite series, potentially considered) can be finite, Greek conceptions distinguished actual and potential infinity; mathematical infinity, following Aristotle himself, was accepted only in potential sense: so it is quite meaningless to suppose any explicit Greek consideration of infinite series. Of course Tacquet’s position, too, must be contextualized: we cannot suppose the presence of our philosophical awareness in 17^{th} century.

2. Focus and methodology: an historical survey

When we introduce historically a mathematical concept, the selection of historical data is relevant (Radford, 1997, p. 28). Problems are connected with their interpretation, which is always based upon cultural institutions and beliefs. Often original data are approached by later editions, so we must take into account editors' conceptions (Barbin, 1994).

We shall not propose a complete survey of the historical roots of the Calculus; we shall present the following references in order to show that: (a) the educational introduction of infinitesimal methods by historical references requires a correct contextualization; (b) historiographically, often an aprioristic platonic epistemological perspective is implicitly assumed.

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3. The exhaustion argument

The exhaustion argument is attributed to Eudoxus (405-355 BC): proofs by exhaustion argument are considered important infinitesimal processes, sometimes proposed in classroom practice.

Proofs by exhaustion argument are based upon the following proposition:

Liber X, Propositio I. Duabus magnitudinibus inequalibus expositis, si à maiori auferatur maius, quàm dimidium,, ab eo, quod eliquum est rursus auferatur maius, quam dimidium;, hic semper fiat: relinquetur tandem quaedam magnitudo, quae minori magnitudine exposita minor erit (Commandino, 1619, p. 123r).

Proposition X-1. Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proved even if the parts subtracted are halves.

Euclid applied the so-called Eudoxus' postulate (which in *Elements* is a *definition*: in III-16 Euclid considered the set of rectilinear and curvilinear angles that is not a class of Archimedean magnitudes; so Greeks were not unaware of quantities that can be infinitesimal):

Liber V, Definitio IV. Proportionem habere inter se magnitudines dicuntur, quae multiplicatae se invicem superate possunt (Commandino, 1619, p. 57v).

Definition V-4. Magnitudes are said to *have a ratio* to one another which can, when multiplied, exceed one another.

Concerning the Proposition X-1, A. Frajese remembers the following fragment by Anaxagoras (500?-428 BC): "For neither is there a least of what is small, but there is always a less. For being isn't non-being" (Frajese, Maccioni, 1970, p. 596).

Can we refer such fragment to the limit notion? Underlying the concept of limit there is the concept of the number system, so it would be necessary to consider the difference between magnitude and number in Greek contribution: the concept of number line is different as seen by Greeks or, for instance, by Cauchy. Hence a direct comparison between Anaxagoras and Cauchy is meaningless.

Let us now consider once again X-1: can we suppose the presence of a limit in the exhaustion argument? M. Kline writes states that "there is no explicit limiting process in it; it rests on the indirect method of proof and in this way avoids the use of a limit" (Kline, 1972, p. 83). The non-equivalence is not only in the formal sense: most differences pertain to the ontological realm (Radford, 2003). In the exhaustion argument we can recognize nowadays an infinitesimal process; but this interpretation is ours, based upon modern conceptions: as Kline notices, the indirect method of proof avoids the use of a limit. Euclid often applied the Proposition X-1, but he neither gave a definition of infinitesimal, nor proposed any particular denomination for infinitesimal processes. Greek beliefs and cultural institutions played a relevant role; Greek way of argumentation was shaped by the social and political context and was developed in the philosophical circles since the 5th century BC (Radford, 1997): such context cannot be forgotten when we interpret Greek contribution.

Let us moreover consider the following propositions:

Liber XII, Propositio I. Symilia polygona, quae in circulis describuntur, inter se sunt, ut

Proposition XII-1. Similar polygons inscribed in circles are to one another as the squares on

diametrorum quadrata (Commandino, 1619, p. 211r).	their diameters.
Liber XII, Propositio II. Circuli inter se sunt ut diametrorum quadrata (Commandino, 1619, p. 211v).	**Proposition XII-2.** Circles are to one another as the squares on their diameters.

Twenty centuries later, G. Saccheri (1667-1733) wrote: “Euclid previously proved (XII-1) that similar polygons inscribed in circles are to one another as the squares on their diameters; by that it would be possible to deduce XII-2, by considering circles as polygons with infinitely many sides” (Saccheri, 1904, p. 104). Saccheri’s remark is interesting, referred to the 17^{th} century, but Greek mathematicians *never* used infinity according to this idea: Euclid’s proof XII-2 is completely different (Frajese, Maccioni, 1970).

4. Towards infinitesimal

In a different context, infinitesimals were considered in a very different way. Cavalieri proposed a new method and a denomination (*indivisibles*) but did not give a definition of indivisible (Lombardo Radice, 1989). Surely his work can be considered a step towards the awareness of infinitesimal concepts; but this judgment is based upon our modern conceptions. Cavalieri’s method, sometimes used in classroom practice, deserves a careful historical introduction.

Cavalieri had no preference for indirect methods (Kline, 1972; *reductio ad absurdum* was used only in Proposition II-12 of *Geometria indivisibilibus continuorom*; and some years later, Cavalieri gave another direct proof of such result in his *Exercitationes geometricae sex*: Lombardo Radice, 1989, p. 256). B. Pascal (1623-1662) and I. Barrow (1630-1677) expressed doubts about the utility of exhaustion argument; P. de Fermat (1601-1665) wrote: “It would be easy to present proofs based upon Archimedean methods; I underline it once and for all, in order to avoid repetitions” (Fermat, 1891-1922, I, p. 257).

Seventeenth-century mathematicians needed effective tools: Cavalieri’s method would not appear completely rigorous (Kline, 1972). But rigor must be investigated in its own conceptual context, in order to avoid the imposition of modern conceptual frameworks to works based upon different ones. It is immensely unlikely that mathematicians in the History could refuse a method because of its foundational weakness that will be pointed out only through a modern approach (Radford, 1997, p. 27).

In fact, frequently historical evaluation is referred to our modern point of view: about J. Wallis, Kline writes: “Wallis, in the *Arithmetica Infinitorum*, advanced the arithmetical concept of the limit of a function as a number approached by the function so that the difference between this number and the function could be made less than any assignable quantity and would vanish ultimately when the process was continued to infinity. His wording is loose but contains the right idea” (Kline, 1972, p. 388). “His wording is loose”: what do we mean by that? If we investigate Wallis’ correctness against our contemporary standards we conclude that his expression is not rigorous. But such investigation would be historically weak: Wallis’ wording would not be correct, *nowadays*; but Wallis *was* rigorous, in his own way.

5. Vanishing quantities

The title of the present section does not suggest a direct comparison between the giants of the mathematics: for instance, I. Newton (1642-1727) and G.W. Leibniz were responsive to his own primary intuition, which in the case of Newton was physical and in the case of Leibniz algebraic.

Leibnizian position was complex: he noticed in 1695 that "a state of transition may be imagined, or one of evanescence" in which "it is passing into such a state that the different is less than any assignable quantity; also that in this state there will still remain some difference, some velocity, some angle, but in each case one that is infinitely small" (in: Kline, 1972, p. 386). "It is apparent that neither Newton nor Leibniz succeeded in making clear, let alone precise, the basic concepts of the Calculus: the derivative and the integral. Not being able to grasp these properly, they relied upon the coherence of the results and the fecundity of the method to push ahead without rigour" (Kline, 1972, p. 387). "It is nevertheless clear that Leibniz allowed himself to be carried away by the very success of his algorithms and was not deterred by uncertainty over concepts" (Boyer, 1985, p. 442). But how can we state "uncertainty over concepts" in Leibnizian thought? We can recognise it through our conceptions; we agree with F. Enriques (1938, p. 60), who pointed out a problem residing into our modern interpretation of Leibnizian ideas.

L. Euler's ideas about infinitesimal are interesting, although a parallelism between Leibniz and Euler cannot be stated uncritically. Euler in his *Institutiones calculi differentialis* (1755) argued:

"Nullum autem est dubium, quin omnis quantitas eousque diminui queat, quoad penitus evanescat, atque in nihilum abeat. Sed quantitas infinite parva nil aliud est nisi quantitas evanescens, ideoque revera erit = 0. Consentit quoque ea infinite parvorum definitio, qua dicuntur omni quantitate assignabili minora: si enim quantitas tam fuerit parva, ut omni quantitate assignabili sit minor, ea certe non poterit non esse nulla; namque nisi esset = 0, quantitas assignari potest ipsi aequalis, quod est contra hypothesin" (Euler, 1787, I, pp. 62-63).

"There is no doubt that every quantity can be diminished to such an extent that it vanishes completely and disappears. But an infinitely small quantity is nothing other than a vanishing quantity and therefore the thing itself equals 0. It is in harmony also with that definition of infinitely small things by which the things are said to be less than any assignable quantity; it certainly would have to be nothing; for unless it is equal to 0, an equal quantity can be assigned to it, which is contrary to the hypothesis" (Kline, 1972, p. 429).

Unfortunately Euler did not see the possibility that a vanishing quantity can be a different kind of quantity from a numerical constant; he was aware of problems with actual infinitesimals, but in mathematical practice he preferred a different approach (Euler, 1796, pp. 84-91). Connections between mathematics and socio-cultural context are fundamental: Euler's approach was not just "tuned in" to applicative features of the scientific frame of mind in the 17^{th} century (Crombie, 1995) and this situation requires a deep study.

6. Necessity of rigor

J.B. d'Alembert's conception about Calculus deserves a careful interpretation; he refused Leibnizian and Eulerian assumptions about differentials and in 1767 stated that a quantity "is something or nothing" and "the supposition that there is an intermediate state between these two is a chimera" (in: Boyer, 1985, p. 493). This point must be considered with reference to d'Alembert's rich personality, linking his Jansenist education with his friendship with Voltaire; moreover it must be seen against the background of the Enlightenment (Grimsley, 1963): "D'Alembert denied the existence of the actually infinite, for he was thinking of geometrical magnitudes rather than of the theory of aggregates proposed a century later. D'Alembert's formulation of the limit concept lacked the clear-cut phraseology necessary to make it acceptable to his contemporaries" (Boyer, 1985, p. 493).

Of course it was impossible for d'Alembert to perceive by intuition ideas introduced by Cantor, so this judgment needs a bit of caution. In his article on *Limit* written for the *Encyclopédie* he stated that one quantity is the limit of a second variable one if the second can approach the first quantity closer than by any assignable quantity, without coinciding with it. This statement is weak if compared with the modern limit notion (Boyer, 1985). But d'Alembert's position must be framed into a socio-cultural context, nearly seventy years before the publication of Cauchy's treatise!

In 1797 J.L. Lagrange tried to reduce Calculus to Algebra (Lagrange, 1813); Kline writes: "Lagrange made the most ambitious attempt to rebuild the foundations of the Calculus. The subtitle of his book reveals his folly. It reads: *Containing the principal theorems of the differential Calculus without the use of the infinitely small, or vanishing quantities, or limits and fluxions, and reduced to the art of algebraic analysis of finite quantities*" (Kline, 1972, p. 430).

Can we consider Lagrange's attempt just as a "folly"? It is hard to forget Kuhn's suggestions quoted at the beginning of the present paper: as a matter of fact, Lagrange's "apparent absurdities" (Kuhn, 1977, p. xii) tried to overcome the weakness of the Calculus. Surely his idea was based upon wrong assumptions (it met great favor for some time, but later it was abandoned: Kline, 1972), nevertheless it must be framed into a wider context: and once again, nowadays, our judgment needs our modern conceptions and our mathematical skill.

7. Concluding remarks

In 1821, A.L. Cauchy gave the following definitions: "When values of a variable approach indefinitely a fixed value, as close as we want, this is the *limit* of all those values. For instance, an irrational number is the limit of the different fractions that gave approximate values of it (…). When values of a variable are (…) lower than any given number, this variable is an *infinitesimal* or an infinitesimal magnitude. The limit of such variable is zero" (Cauchy, 1884-1897, p. 4).

Cauchy introduced the distinction between *constants* and *variable quantities*, although he had no formal axiomatic description of real numbers. It is educationally interesting to underline that Cauchy's verbal formulation was expressed in the paradigm available at the time: nowadays it can lead to the use of different representation registers.

As noticed, presented examples are not a full collection of historical data referred to the limit notion: many authors are still missing, e.g. L. Valerio, K.T.W. Weierstrass, A. Robinson (1966). For instance, Weierstrass' definition of limit allows a modern symbolic representation, although it would be misleading to make reference to a single symbolic register: there are different registers in different communities of practice; Leibniz, Newton, Cauchy had their own symbolic registers which differ from each other and differ, too, from that of Weierstrass (Bagni, to appear-a).

The passage from discrete to continuum is a cultural problem and many historical references are important in order to approach it. Turning back to educational issues, the transfer of some situations from History to Didactics needs a wider cultural dimension keeping into account non-mathematical elements, too (Radford, 1997). Some experimental results seem to suggest that in classroom practice we can see, in students' minds, several reactions, doubts and inner representations that we can find in the History (Tall, Vinner, 1981; Bagni, 2005); but many aspects must be considered: for instance, what do we mean by "students' minds"? Can we consider one's mind as a "mirror of nature" (Rorty, 1979) and make reference to our "inner representations" uncritically? According to W.V.O. Quine, "epistemology, or something like it, simply falls into place as a chapter of psychology and hence of natural science. It studies a natural phenomenon, viz., a particular human subject" (Quine, 1969, p. 82). And R. Rorty highlights the importance of "the community as source of epistemic authority" (Rorty, 1979, p. 380): "We need to turn outward rather then inward, toward the social context of justification rather than to the relations between inner representations" (Rorty, 1979, p. 424).

Of course this theoretical perspective needs further research in order to be effectively applied in educational practice (see for instance: Bagni, D'Amore, 2005; Bagni, to appear-b). Nevertheless we can state that a sociological approach is very important: as a matter of fact, a crude paralleling of History with learning processes would connect two cultures referring to quite different contexts (Radford, 1997), so it cannot be proposed without a clear consideration of the social and cultural backgrounds. An "internalist" History, so a conception of the development of mathematics as a pure subject, isolated from "external" influences, is hardly useful in education (Grugnetti, Rogers, 2000, p. 40).

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