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INEQUALITIES AND EQUATIONS: HISTORY AND DIDACTICS

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Abstract. *The historical development of equations and inequalities is examined, in order to underline their very different roles in various socio-cultural contexts. From the educational point of view, historical differences must be adequately taken into account: as a matter of fact, a forced analogy between equations and inequalities, in procedural sense, would cause some dangerous phenomena.*

1. Introduction: algebraic equations and inequalities

Frequently, from the educational point of view, algebraic inequalities are introduced to pupils after algebraic equations, and the solving techniques are strictly compared; nevertheless, in classroom practice, techniques for equation solving, when applied to inequalities, lead sometimes to wrong results: so didactic connections between equations and inequalities are not simple to be stated (a number of papers can be found; for instance: Linchevski & Sfard, 1991 and 1992; Fischbein & Barash, 1993; Tsamir, Tirosh & Almog, 1998). Some experimental studies by L. Bazzini and P. Tsamir (2002) clearly pointed out several meaningful situations.

Let us note that the word *equation*, in English, denotes the mathematical statement of an *equality*. For instance, by writing "$x+2 = 5$" (*equation*) we state that the $x+2$ is *equal* to 5: and this is true if and only if $x = 3$ (solution of the considered equation). Of course we can consider an equality also without a proper equation, e.g. without an unknown: when we write, for instance, "$2+7 = 9$" we state that the sum of the numbers 2 and 7 is equal to 9 (frequently a statement of an equality that is true for all values of a variable, e.g. "$2x+7x = 9x$", is indicated by the word *identity*) and this is true. From the logical point of view, "$2+7 = 9$" is a sentence that expresses a proposition with the truth value "true"; "$x+2 = 5$" is not a sentence: it does not express a proposition, but a condition regarding the values which may be assigned to the variable involved (Bell & Machover, 1977, p. 12) and it will assume a truth value, either "true" or "false", depending on which number is assigned to x as a value.

Let us now consider the *inequality* "$x+2 < 5$": by that we state that $x+2$ is *less than* 5 and this is true if and only if $x < 3$. In several languages the word *inequality* can assume two different versions, so it is translated by two different words: for instance, in French, these words are *inégalité* (in Italian: *disuguaglianza*) and *inéquation* (*disequazione*).[1] With reference to these words, the mentioned difference would be summarised as follows: an *inéquation* is the mathematical statement of an *inégalité*.

Both from a logical point of view and from an educational point of view, there is a great difference between an inequality like "$x+2 < 3$" and an inequality like $1+2 < 5$: their epistemological status is clearly different.

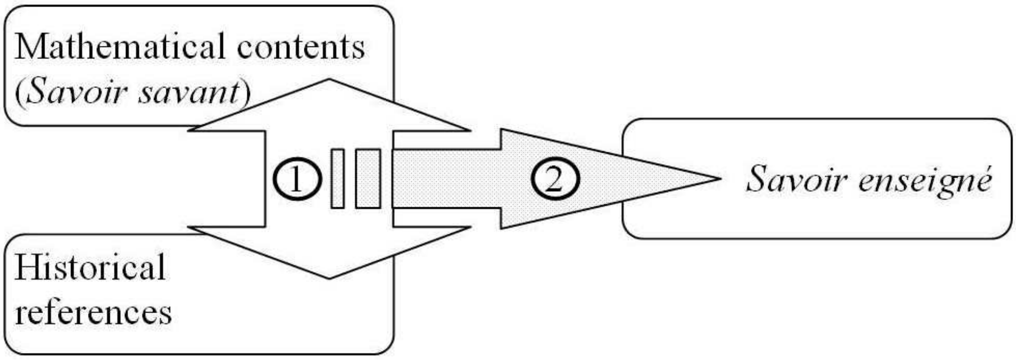
[1] Sometimes, in English, an *inequation* denotes a statement that two quantities or expressions are *not the same*, or *do not represent the same value* (written by a crossed-out equal sign: $x \neq y$).

We shall denote the first inequality by the term *inéquation*, the second by *inégalité*.

2. History and didactics: different theoretical perspectives

Our work will take into account some references from the history of Algebra. As several studies have pointed out, the historical approach can play a valuable role in mathematics teaching and learning and it is a major issue of the research in mathematics education, with reference to all school levels (Heiede, 1996).

The use of the history into education links psychological learning processes with historical-epistemological issues (Radford, Boero & Vasco, 2000, p. 162) and this link is ensured by epistemology (Moreno & Waldegg, 1993). Concerning the features of interactions between history and educational practice, a wide range of views can be examined. Different levels can be considered with reference to teaching-learning processes: a first is related to anecdotes presentation (and it can be useful in order to strengthen pupils' conviction: Radford, 1997); higher levels bring out metacognitive and multidisciplinary possibilities. Let us consider the following representation:



(where some well known terms by Y. Chevallard are employed). Of course this is just a schematic outline: for instance, the passage from the *savoir savant* to the *savoir enseigné* is not simple. However two sets of connections must be analysed:

- connections (1) between mathematical contents and historical references;
- connections (2) between mathematical contents linked to historical references and knowledge presented to pupils in classroom (after the *transposition didactique*).

Different uses of the history into didactics do not reflect just practical educational issues: they imply different epistemological assumptions (Radford, 1997 and 2003). For instance, the selection of historical data to be presented in classroom practice is epistemologically relevant: this selection reflects some epistemological choices by the teacher, too. Important problems are related to the interpretation of historical data: this is frequently based upon our cultural institutions and beliefs (Gadamer, 1975).

Frequently the role of the history into didactics is considered from an introductory point of view[2]: sometimes a parallelism between the historical development and the

[2] Teachers can be induced to apply historical knowledge to classroom practice according to a naïve approach (as noted in Radford, 1997): for instance the educational introduction of a topic would take place just by the ordered presentation of all the historical references related with it.

cognitive growth is assumed (since E. Haeckel's "law of recapitulation", 1874; see: Piaget & Garcia, 1989). As a matter of fact, a new concept is often encountered by mathematicians in operative stages, for instance in problem solving activities, and it will be theoretically framed many years or several centuries later (Furinghetti & Radford, 2002); a parallel evolution can be pointed out in the educational field: often the first contact with a new notion takes place in operative stages (Sfard, 1991; see the discussion in: Radford, 1997): in fact, pupils' reactions are sometimes rather similar to reactions noted in mathematicians in history (Tall & Vinner, 1981) and such correspondence would be an important tool for mathematics teachers.

The mentioned parallelism would require a theoretical framework: as a matter of fact, it leads to epistemological issues. A major issue is related to the interpretation of history: for instance, is it correct to present the history as a path that, by unavoidable mistakes, obstacles overcoming and critical reprises, finally leads to our modern theories? What is the role played by social and cultural factors that influenced historical periods? Mathematical contents deal with non-mathematical context, too, and knowledge must be understood in terms of cultural institutions (Bagni, 2004).

According to the "epistemological obstacles" perspective by G. Brousseau, one of the most important goals of historical studies is finding problems and systems of constraints (*situations fondamentales*) that must be analysed in order to understand existing knowledge, whose discovery is connected to the solution of such problems (Brousseau, 1983; Radford, Boero & Vasco 2000, p. 163). Obstacles are subdivided into epistemological, ontogenetic, didactic and cultural ones (Brousseau, 1989) and this subdivision points out that the sphere of the knowledge is considered isolate from other spheres. This perspective is characterised by other important assumptions (Radford, 1997): the reappearance in teaching-learning processes, nowadays, of the same obstacles encountered by mathematicians in the history; and the exclusive, isolated approach of the pupil to the knowledge, without taking into account social interactions with other pupils and teachers.

With reference to the above-presented schematic picture, we can summarise epistemological assumptions as follows:

(1) knowledge exists and represents the best solution of relevant problems; epistemological obstacles recur either in history or in educational practice;

(2) the sphere of knowledge is separated from educational and cultural spheres; pupils approach knowledge individually.

The crucial point is the following (Gadamer, 1975): is it possible, nowadays, to see historical events without the influence of our modern conceptions? As a matter of fact, we can explicitly accept the presence of our modern point of view: in other words, we can take into account that, when we look at the past, we connect two cultures that are "different [but] they are not incommensurable" (Radford, Boero & Vasco, 2000, p. 165). Concerning the nature of mathematics, "the historical approach encourages and enables us to regard mathematics not as a static product, with *a priori* existence, but as an intellectual process; not as a complete structure dissociated from the world, but as an on-going activity of individuals” (Grugnetti & Rogers, 2000, p. 45; see also the “voices and echoes” perspective: Boero & Al. 1997).

According to the socio-cultural perspective by L. Radford, knowledge is linked to activities of individuals and, as we above noted, this is strictly related to cultural institutions; knowledge is not built individually, but into a wider social context (Radford, Boero & Vasco, 2000, p. 164). The role played by the history must be interpreted with reference to different socio-cultural situations (Radford, 2003) and it gives us the opportunity for a deep critical study of considered historical periods.

With reference to the above-presented picture, we can summarize two different epistemological assumptions from the previous ones as follows:

(1) knowledge is related to actions required in order to solve problems; problems are solved within the socio-cultural contexts of the considered periods;

(2) knowledge is socially constructed; cultural institutions and beliefs of their own culture influence pupils.

3. The selection of historical data: the history of algebraic notation

We previously stated that the selection of historical data is epistemologically relevant to the historical introduction of a concept. A classical example (Radford, 1996 and 1997) is relevant to our research.

In 1842, G.H.F. Nesselmann characterised three main stages in the historical development of algebraic notation (see: Serfati, 1997):

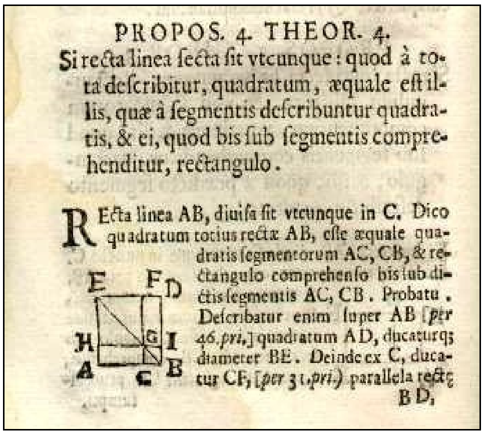
Rhetorical Algebra	***(from) Words***
(Egyptians, Babylonians etc.)	
Syncopated Algebra	⇩
(Pacioli, Cardan etc.)	
Symbolic Algebra	
(Descartes etc.)	***(to) Symbols***

(concerning rhetorical algebra, original Nesselman’s approach would be referred to Arabs: the interpretation of Babylonian mathematical texts is not so ancient).

This sequence can suggest a progressive elimination of non-mathematical verbal expressions: mathematical objects would be “purified by taking away all their insane physical substance” (Radford, 1997, p. 28); it suggests the existence of a definitive algebraic language, so that the historical development is the progressive approaching to our modern, pure expression. But this traditional summary can be considered as a full expression of the history of algebraic language? Important steps are still missing: for instance, we must remember the Greek “Geometric Algebra” (this denomination was given by H.G. Zeuthen, with reference to the 2^{nd} Book of Eulclid’s *Elements*) and the symbolism introduced by Diophantus of Alexandria (3^{rd}-4^{th} centuries).

Roots of the “Geometric Algebra” are related to Eudoxus of Cnidus (408-355 B.C.) who introduced the notion of a magnitude standing for entities such as line segments, areas, volumes (Kline, 1972, p. 48). No quantitative values were assigned to such magnitudes (so Eudoxian ideas avoid irrational numbers as numbers) and this allowed Greeks to give general results: the figure is referred to the 4th Proposition of the 2nd Books of *Elements*.

Nowadays this proposition is expressed by: $(a+b)^2 = a^2+b^2+2ab$, but in *Elements* only the picture gives the proof of this statement.


PROPOS. 4. THEOR. 4.

Si recta linea ſecta ſit vtcunque: quod à tota deſcribitur, quadratum, æquale eſt illis, quæ à ſegmentis deſcribuntur quadratis, & ei, quod bis ſub ſegmentis comprehenditur, rectangulo.

Recta linea AB, diuiſa ſit vtcunque in C. Dico quadratum totius rectæ AB, eſſe æquale quadratis ſegmentorum AC, CB, & rectangulo comprehenſo bis ſub dictis ſegmentis AC, CB. Probatu. Deſcribatur enim ſuper AB [*per 46. pri.*] quadratum AD, ducaturq; diameter BE. Deinde ex C, ducatur CF, [*per 31. pri.*] parallela rectæ BD,

(Rondelli, G.: 1693, *Euclidis Elementa*, Longo, Bologna, p. 80)

Six centuries later, Diophantus of Alexandria introduced an algebraic symbolism, and this is “one of Diophantus’ major steps” (Kline, 1972, p. 139).[3] This symbolism is complicated and it is not complete (the main difference between Diophantine symbolism and our modern algebraic notation is the lack of symbols for operations and relations: Boyer, 1985, p. 202); Diophantine Algebra has been called *syncopated* (see: Boyer, 1985, p. 201; Kline, 1972, p. 140), but if we compare Diophantus’ syncopation and, for instance, Cardan’s one we realise that they are very different: Diophantus obtained fundamental achievements (Greek Algebra “no longer was restricted to the first three powers or dimensions”: Boyer, 1985, p. 202), while European syncopated Algebra (15th-16th centuries) seems to be “a mere technical strategy that the limitations of writing and the lacks of printing in past times imposed on the diligent scribes that had to copy manuscripts by hand” (Radford, 1997, p. 29).

If we rewrite our summary taking into account those new elements, we have:

Rhetorical Algebra (Egyptians, Babylonians etc.)	***Words***
Greek “Geometric Algebra”	***Pictures***
Diophantus of Alexandria	*Incomplete symbolism (?)*
Syncopated Algebra (Pacioli, Cardan etc.)	*Abbreviated words (?)*
Symbolic Algebra (Descartes etc.)	***Symbols***

So how can we describe the history of Algebra only in the sense of a progressive “purification”, if we consider Geometric Algebra and Diophantine symbols?

4. From history to didactics: equations and inequalities

Previous discussion underlines that algebraic processes have not been expressed by symbols for a long time, but the evolution of algebraic notation does not reflect just the progressive elimination of “insane physical substance” (Radford, 1997, p. 28).

[3] Some Diophantine symbols appear in a collection of problems probably antedating Diophantus’ *Arithmetica* (as noted in: Boyer, 1985, p. 204; Robbins, 1929).

Several elements must be taken into account: for instance, it is important to point out that mathematical expression was initially oral. More generally, relevant non-mathematical elements must be considered: the development of western mathematical symbolism is to be framed into the correct cultural context, towards a systematization of human expression.

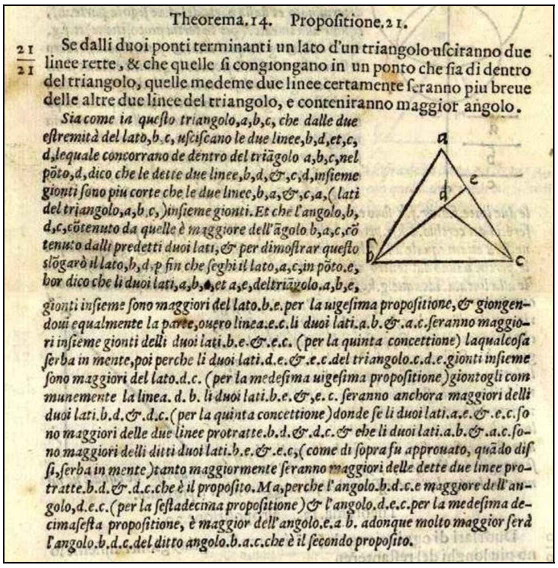
Historical evolution is complex: for instance, G. Lakoff and R. Núñez note: "It may be hard to believe, but for two millennia, up to the 16^{th} century, mathematicians got by without a symbol for equality" (Lakoff & Núñez, 2000, p. 376). Of course the role of "=" cannot be considered too simple: "Even an idea as apparently simple as equality involves considerable cognitive complexity. […] An understanding of what "=" means requires a cognitive analysis of the mathematical ideas involved" (Lakoff & Núñez, 2000, p. 377; Arzarello, 2000). In the first paragraph we noted several differences between equalities and equations, and other important differences can be mentioned (see: Lakoff & Núñez, 2000, p. 376).

Let us now sketch some historical references regarding equation and inequalities.

The history of equations is rich and different mathematical cultures in many part of the world dealt with processes that can be related to equations; in the Renaissance, the so-called *Regola d'Algebra* (algebraic rule) was the process for arithmetic problem solving based upon the resolution of an algebraic equation (Franci & Toti Rigatelli, 1979, p. 7).

As we shall see, the history of inequalities is not so rich. Ancient inequalities, too, were expressed by verbal registers; it is important to underline that an inequality (see the picture, referred to a geometric inequality dealing with 21^{st} Proposition of the 1^{st} Book of *Elements*) is often only the expression of an *inégalité*.

Some inequalities in the proper sense of *inéquation* can be related to the development of the Calculus, e.g. to majorizing/minorizing (see: Hairer & Wanner, 1996).[4] Let us now consider some texts published in 19^{th} century; two treatises by P. Ruffini (1765-1822) were included in the 3^{rd}-5^{th} parts of *Corso di Matematiche* (Modena, Italy, 1806 and 1808).



Theorema.14. Propoſitione.21.

21/21 Se dalli duoi ponti terminanti un lato d'un triangolo uſciranno due linee rette, & che quelle ſi congiongano in un ponto che ſia di dentro del triangolo, quelle medeme due linee certamente ſeranno piu breue delle altre due linee del triangolo, e conteniranno maggior angolo.

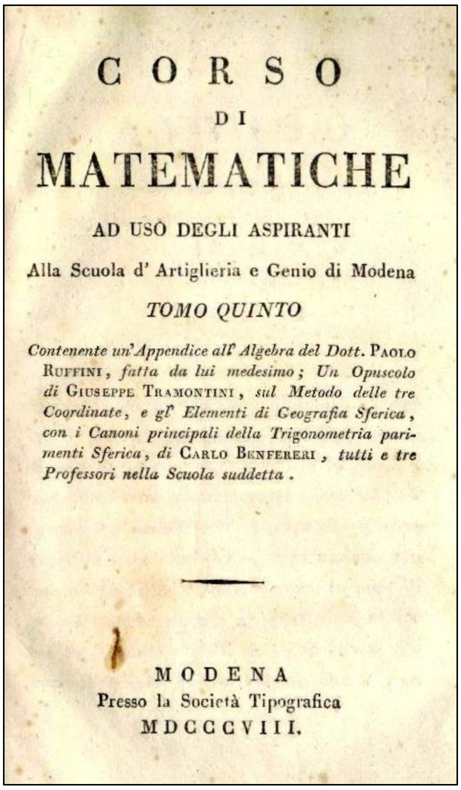
Sia come in queſto triangolo,a,b,c, che dalle due eſtremità del lato,b.c, uſciſcano le due linee,b,d,et,c,d,lequale concorrano de dentro del triāgolo a,b,c,nel pōto,d,dico che le dette due linee,b,d,&,c,d,inſieme gionti ſono piu corte che le due linee,b,a,&,c,a,(lati del triangolo,a,b,c,)inſieme gionti. Et che l'angolo,b,d,c,cōtenuto da quelle è maggiore dell'āgolo b,a,c,cō tenuto dalli predetti duoi lati,& per dimoſtrar queſto slōgarò il lato,b,d,p fin che ſeghi il lato,a,c,in pōto.e, hor dico che li duoi lati,a,b,et a,e,del triāgolo.a,b,e, gionti inſieme ſono maggiori del lato.b.e.per la uigeſima propoſitione,& giongendoui equalmente la parte,ouero linea.e.c.li duoi lati.a.b.&.a.c.ſeranno maggiori inſieme gionti delli duoi lati.b.e.&.e.c. (per la quinta concettione) laqualcoſa ſerba in mente,poi perche li duoi lati.d.e.&.e.c.del triangolo.c.d.e.gionti inſieme ſono maggiori del lato.d.c. (per la medeſima uigeſima propoſitione)giontogli communemente la linea. d. b. li duoi lati.b.e.&,e. c. ſeranno anchora maggiori delli duoi lati.b.d.&.d.c.(per la quinta concettione)donde ſe li duoi lati.a.e.&.e.c.ſono maggiori delle due linee protratte.b.d.&.d.c.& che li duoi lati.a.b.&.a.c.ſono maggiori delli ditti duoi lati.b.e.&.e.c,(come di ſopra fu approuato, quādo diſſi,ſerba in mente)tanto maggiormente ſeranno maggiori delle dette due linee protratte.b.d.&.d.c.che è il propoſito.Ma,perche l'angolo.b.d.c.e maggiore dell'angolo,d.e.c.(per la ſeſtadecima propoſitione)& l'angolo.d.e.c.per la medeſima decimaſeſta propoſitione, è maggior dell'angolo.e.a b. adonque molto maggior ſerà l'angolo.b.d.c.del ditto angolo.b.a.c.che è il ſecondo propoſito.

(Tartaglia, N., 1569: *Euclide Megarense*, Bariletto, Venezia, p. 27)

Let us propose some quotations:

[4] We cannot forget the well known statement by J. Dieudonné in the *Préface* of his *Calcul infinitesimal* (Hermann, Paris 1980): "En d'autres termes, le Calcul infinitésimal, tel qu'il se présente dans ce livre est l'apprentissage de maniement des *inégalités* bien plus que des égalités, et on pourrait le résumer en trois mots: *majorer, minorer, approcher*".

- in the 3rd vol. (*Algebra*), p. 24, a property of equivalence for equations is explicitly stated: "Given the equation A–B–C = –D+E, we can carry the terms from the first to the second member and from the second to the first member, and we shall have: D–E = –A+B+C" (the translation is ours); it is important to underline that in the considered treatise no similar properties are stated with reference to inequalities;
- in the 3rd vol., p. 146, inequalities are proposed and solved in order to express some particular conditions for the solutions of some given equations. Frequently examples deal with similar conditions (in the 5th vol., *Appendice all'Algebra*, too): so inequalities are often combined to equations and to simultaneous equations in order to express some conditions.



CORSO
DI
MATEMATICHE
AD USO DEGLI ASPIRANTI
Alla Scuola d' Artiglieria e Genio di Modena
TOMO QUINTO
Contenente un'Appendice all' Algebra del Dott. PAOLO RUFFINI, *fatta da lui medesimo; Un Opuscolo di* GIUSEPPE TRAMONTINI, *sul Metodo delle tre Coordinate, e gl' Elementi di Geografia Sferica, con i Canoni principali della Trigonometria parimenti Sferica, di* CARLO BENFERERI, *tutti e tre Professori nella Scuola suddetta.*
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Moreover, an interesting quotation can be considered with reference to the 20th century. P. Odifreddi writes: "A contribution by von Neumann was the solution, in 1937, of a problem posed by L. Walras in 1874. […] He noted that a model must be expressed by inequalities (as we usually do nowadays) and must not be expressed just by equations (as mathematicians were accustomed to do in that period), then he found a solution by Brouwer's theorem".[5]

So we can point out an interesting historical asymmetry: mathematicians usually expressed the problem to be solved by equations (Franci & Toti Rigatelli, 1979, p. 7); then, by inequalities (in the proper sense of *inéquation*), they expressed some conditions for the solutions of the considered equations. Moreover, in the history, the resolution of an inequality (*inéquation*) has been often obtained by solving an equation that practically replaced the assigned inequality. Social and cultural contexts must be taken into account: frequently the "practical solution" has been considered the main result to be obtained, much more important than the "field of possibilities". So a meaningful social importance has been attributed to the process by which the solution can be obtained (see the use of practical methods in order to improve the precision of the solutions: Hairer & Wanner, 1996).

5. Final reflections

Although recently the autonomous role of inequalities (in the sense of *inéquation*, too) has been educationally recognised, in classroom practice there is still an operative dependence, a relevant "subordination". For instance, an inequality

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[5] Quotation from the website www.matematicamente.it/articoli; the translation is ours.

characterises a subset of the set of real numbers, frequently an infinite subset, a segment or a half-line. Main features of these subset are sometimes their “boundary points” (for instance, the ends of the segment): and they can be obtained by solving the equation obtained by replacing “<” with “=” in the given inequality.

Important metaphors (related to Arithmetics) are based upon “physical segments”: for instance, we can propose the correspondence of a number with a distance that “can be measured by placing physical segments of unit length end-to-end and counting them” (Lakoff & Núñez, 2000, p. 68). Moreover: “When we move in a straight line from one place to another, the path of our motion forms a physical segment […]. There is a simple relationship between a path of motion and a physical segment. The origin of the motion corresponds to one end of a physical segment, the endpoint of the motion corresponds to the other end of the physical segment; and the path of motion corresponds to the rest of the physical segment” (Lakoff & Núñez, 2000, pp. 71-72). So in the framework of the *embodied cognition* the physical description of a segment (and of an half-line) has its origin in an end of the segment and its endpoint in the other end (in the case of an half-line, it goes on indefinitely): this underlines once again the importance of mentioned “boundary points”.

Frequently the first step (and, in many cases, it is the main step) of the resolution of an algebraic inequality (*inéquation*) is the resolution of an equation: as a matter of fact, in order to solve $a(x) < b(x)$ we must solve the equation $a(x) = b(x)$. So a forced but sometimes improper educational analogy can be considered, besides the historical asymmetry. This can cause some dangerous phenomena, in procedural sense: in order to avoid breaks between sense and denotation of algebraic expressions, L. Bazzini and P. Tsamir suggest a functional approach, an integrated introduction of equations and inequalities based upon the concept of function (Bazzini & Tsamir, 2002). The use of historical references, correctly considered in their own contexts, can help us to present the different roles and to underline the procedural differences between equations and inequalities.

Let us finally note that the 2nd order knowledge (we make reference to: Drouhard & Panizza, 2003) is relevant to the correct educational presentation of equations and inequalities. For instance, concerning the semiotic representation, the forced analogy between algebraic equations and inequalities, implicitly considered in their sequential presentation, can be referred also to employed representation registers: as a matter of fact, symbolic registers can suggest similar operative approaches to $f(x) = g(x)$ and to $f(x) < g(x)$. So the use of non-symbolic registers (for instance the visual register, directly involved in the functional approach) can be useful; of course the co-ordination of employed register is a very important point (Duval, 1995).

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