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MATHEMATICS EDUCATION AND HISTORICAL REFERENCES: GUIDO GRANDI'S INFINITE SERIES

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Abstract. Integrating history of mathematics into the mathematics education is an important point: the use of history in education can be an effective tool for the teacher. In this paper an example from the history of mathematics (Grandi's infinite series, 1703, and Leibniz remarks, 1716) is presented and its educational utility is investigated. Students' behaviour is examined, with reference to pupils aged 16-18 years: we conclude that historical examples are useful in order to improve teaching of infinite series, e.g. to stimulate reflections about actual and potential infinity. Nevertheless the main problem of the passage from finite to infinite is a cultural one, and historical issues are important in order to approach it, although the historical approach can be considered together with other educational approaches.

Key words: actual and potential infinity, didactical contract, history of mathematics, infinite series, probabilistic argument, socio-cultural context.

1. HISTORY AND MATHEMATICS EDUCATION

> "Even 500 years ago a philosophy of mathematics was possible, a philosophy of what mathematics was then".
>
> Ludwig Wittgenstein (1956, IV, 53)

Several theoretical frameworks can be mentioned in order to link learning processes with historical issues (Fauvel & van Maanen, 2000; Cantoral & Farfán, 2004, see in particular the Chapter 8). According to the "epistemological obstacles" perspective (Bachelard, 1938; Brousseau, 1983), one of the main goals of historical study is finding systems of constraints (the so-called *situations fondamentales*) that must be studied in order to understand knowledge, whose discovery is connected to their solution. Some Authors (Radford, Boero & Vasco 2000, p. 163) notice that this perspective is characterised by an important assumption: the reappearance in our teaching-learning processes, in the present, of the obstacles encountered by mathematicians in the past. Nevertheless, historical data must be considered nowadays and several issues are connected with their interpretation, based upon our cultural institutions and beliefs; according to Luis Radford's socio-cultural perspective, knowledge is linked to activities of individuals and this is essentially related to cultural institutions (Radford, 1997); knowledge is not built individually, but in a wider social context (Radford, Boero &

Vasco, 2000, p. 164). So the *savoir savant* (Chevallard, 1985) cannot be considered absolute and it must be understood in terms of cultural institutions (Lizcano, 1993; Barbin, 1994; Grugnetti & Rogers, 2000; Dauben & Scriba, 2002).

A first notion of infinite series may well have a very ancient source: Aristotle of Stagira (384-322 BC) implicitly underlined that the sum of a series of infinitely many addends (potentially considered) can be a finite quantity (*Physics*, III, VI, 206 b, 1-33). In his *Quadratura parabolae*, Archimedes of Syracuse (287-212 BC) considered implicitly a geometric series. Several centuries later, Andreas Tacquet (1612-1660) noticed that the passage from a "finite progression" to an infinite series would be "immediate" (Loria, 1929-1933, p. 517); but such a passage is crucial:[1] Greek conceptions strictly distinguished actual and potential infinity (and mathematical infinity, following Aristotle, was accepted only in a potential sense).[2]

In this work we shall discuss some educational aspects related to infinite series by using historical examples (see for instance: Edwards, 1994; Hairer & Wanner, 1996; Bagni, 2000a). In particular, we are going to discuss two main points:

- can we consider effectively a parallelism between students' justifications and some examples from the history of mathematics? And does this bear out the well-known idea by Piaget and Garcia (see: Piaget & Garcia, 1983), according to which historical development and individual development are linked?
- moreover, is our students' approach to infinite series influenced by potential infinity or by actual infinity?

We shall consider a brief historical survey and a case study,[3] in order to propose some hints for further research.

2. INFINITE SERIES: GRANDI, LEIBNIZ, RICCATI

2.1. Grandi and Leibniz

We are going to examine a well-known indeterminate series. In 1703, Guido Grandi (1671-1742) noticed that from $1-1+1-1+\ldots$ it is possible to obtain 0 or 1:

$$(1-1)+(1-1)+(1-1)+(1-1)+\ldots = 0+0+0+0+\ldots = 0$$
$$1+(-1+1)+(-1+1)+(-1+1)+\ldots = 1+0+0+0+\ldots = 1$$

The sum of the alternating series = $1-1+1-1+\ldots$ was considered ½ by Grandi.[4] According to him, the proof can be based upon the following expansion (now expressed using modern notation), nowadays accepted if and only if $|x| < 1$:

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[1] Of course it is possible to employ several visual representations (see for instance: Duval, 1995; Bagni, forthcoming-a and forthcoming-b).

[2] In fact, Tacquet made reference to ancient mathematics without any historical contextualization. His position, too, must be contextualised: we cannot suppose the presence of our philosophical awareness in the 17^{th} century (of course it is necessary to take into account both the period in which a work was written and the period of its edition or comment: Barbin, 1994; Dauben & Scriba, 2002).

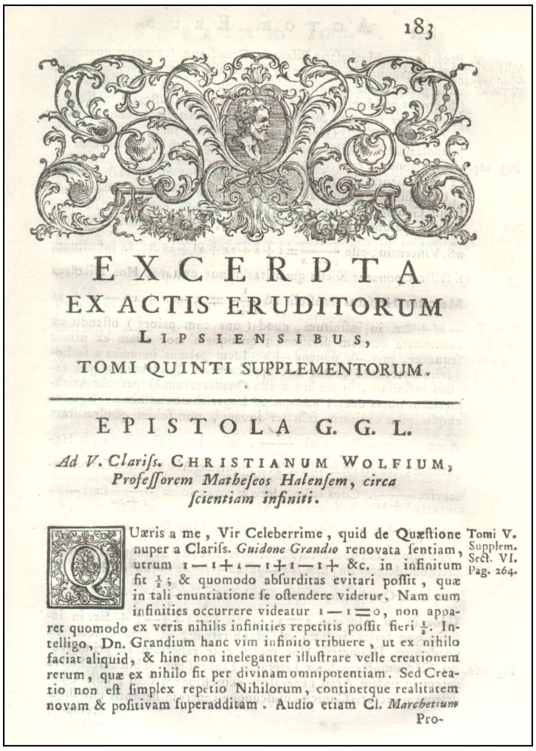
[3] Some experimental data discussed in this paper are reprised by: Bagni, 2005.

[4] Euler and Fourier also thought that $1-1+1-1+\ldots = ½$.

$$\frac{1}{1+x} = \sum_{i=0}^{+\infty} (-x)^i = 1 - x + x^2 - x^3 + \ldots$$

From $x = 1$ (of course this is *not* correct)[5] we should have: 1–1+1–1+... = ½.

Gottfried Wilhelm Leibniz (1646-1716) studied Grandi's series and in his letter to Jacopo Riccati (1676-1754) probably written in 1715, stated that Grandi's solution is correct ("In *Acta Eruditorum Lipsiae* I think I have solved this problem": Michieli, 1943, p. 579).[6] Moreover, Leibniz studied Grandi's series in some letters (1713-1716) to German philosopher Christian Wolf (1678-1754),[7] where he introduced the "probabilistic argument" (that influenced, for instance, Johann and Daniel Bernoulli).



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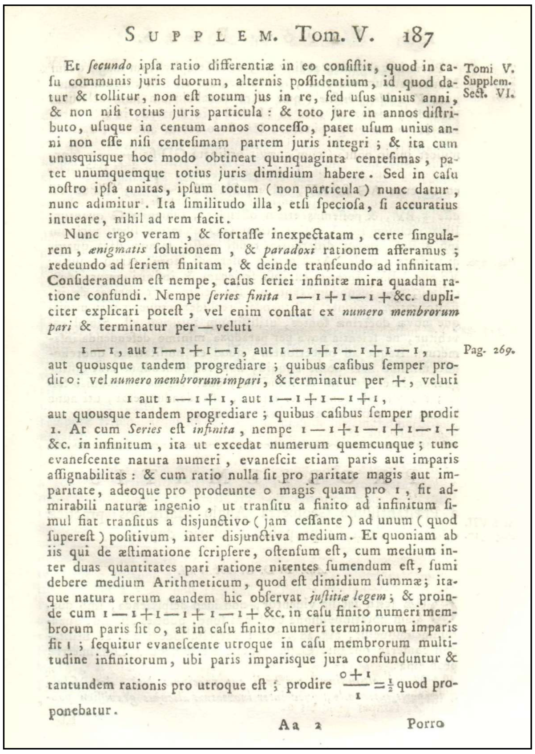
EXCERPTA
EX ACTIS ERUDITORUM
LIPSIENSIBUS,
TOMI QUINTI SUPPLEMENTORUM.

EPISTOLA G. G. L.

Ad V. Clariſs. CHRISTIANUM WOLFIUM, *Profeſſorem Matheſeos Halenſem, circa ſcientiam infiniti.*

QUæris a me, Vir Celeberrime, quid de Quæſtione nuper a Clariſs. *Guidone Grandio* renovata ſentiam, utrum 1—1+1—1+1—1+ &c. in infinitum fit ½; & quomodo abſurditas evitari poſſit, quæ in tali enuntiatione ſe oſtendere videtur. Nam cum infinities occurrere videatur 1—1=0, non apparet quomodo ex veris nihilis infinities repetitis poſſit fieri ½. Intelligo, Dn. Grandium hanc vim infinito tribuere, ut ex nihilo faciat aliquid, & hinc non ineleganter illuſtrare velle creationem rerum, quæ ex nihilo fit per divinam omnipotentiam. Sed Creatio non eſt ſimplex repetio Nihilorum, continetque realitatem novam & poſitivam ſuperadditam. Audio etiam Cl. *Marchettum* Pro-

Tomi V. Supplem. Sect. VI. Pag. 264.



SUPPLEM. Tom. V. 187

Et *ſecundo* ipſa ratio differentiæ in eo conſiſtit, quod in caſu communis juris duorum, alternis poſſidentium, id quod datur & tollitur, non eſt totum jus in re, ſed uſus unius anni, & non niſi totius juris particula: & toto jure in annos diſtributo, uſuque in centum annos conceſſo, patet uſum unius anni non eſſe niſi centeſimam partem juris integri; & ita cum unusquisque hoc modo obtineat quinquaginta centeſimas, patet unumquemque totius juris dimidium habere. Sed in caſu noſtro ipſa unitas, ipſum totum (non particula) nunc datur, nunc adimitur. Ita ſimilitudo illa, etſi ſpecioſa, ſi accuratius intueare, nihil ad rem facit.

Tomi V. Supplem. Sect. VI.

Nunc ergo veram, & fortaſſe inexpectatam, certe ſingularem, *ænigmatis* ſolutionem, & *paradoxi* rationem afferamus; redeundo ad ſeriem finitam, & deinde tranſeundo ad infinitam. Conſiderandum eſt nempe, caſus ſeriei infinitæ mira quadam ratione confundi. Nempe *ſeries finita* 1—1+1—1+&c. dupliciter explicari poteſt, vel enim conſtat ex *numero membrorum pari* & terminatur per — veluti

1—1, aut 1—1+1—1, aut 1—1+1—1+1—1, Pag. 269.

aut quousque tandem progrediare; quibus caſibus ſemper prodit 0: vel *numero membrorum impari*, & terminatur per +, veluti

1 aut 1—1+1, aut 1—1+1—1+1,

aut quousque tandem progrediare; quibus caſibus ſemper prodit 1. At cum *Series* eſt *infinita*, nempe 1—1+1—1+1—1+ &c. in infinitum, ita ut excedat numerum quemcunque; tunc evaneſcente natura numeri, evaneſcit etiam paris aut imparis aſſignabilitas: & cum ratio nulla ſit pro paritate magis aut imparitate, adeoque pro prodeunte 0 magis quam pro 1, fit admirabili naturæ ingenio, ut tranſitu a finito ad infinitum ſimul fiat tranſitus a disjunctivo (jam ceſſante) ad unum (quod ſupereſt) poſitivum, inter disjunctiva medium. Et quoniam ab iis qui de æſtimatione ſcripſere, oſtenſum eſt, cum medium inter duas quantitates pari ratione nitentes ſumendum eſt, ſumi debere medium Arithmeticum, quod eſt dimidium ſummæ; itaque natura rerum eandem hic obſervat *juſtitiæ legem*; & proinde cum 1—1+1—1+1—1+ &c. in caſu finito numeri membrorum paris ſit 0, at in caſu finito numeri terminorum imparis ſit 1; ſequitur evaneſcente utroque in caſu membrorum multitudine infinitorum, ubi paris imparisque jura confunduntur & tantundem rationis pro utroque eſt; prodire $\frac{0+1}{1} = \frac{1}{2}$ quod proponebatur.

Aa 2 Porro

Let us see more precisely Leibnitian argument dealing with the question

"if 1–1+1–1+1–1+&c. *in infinitum* is ½ and how we can avoid the absurdity that in this statement can be recognised. As a matter of fact, when we consider an infinity of

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[5] From an educational point of view, it can be noticed that this (wrong) result can be achieved by the (wrong) procedure: from s = 1–1+1–1+... we should have: s = 1–(1–1+1–...) and s = 1–s, so s = ½. Of course, nowadays, this procedure cannot be accepted: it is clearly based upon a quite incorrect use of arithmetical properties and upon the implicit statement that 1–1+1–1+... is a number s, and we know that this is false (Bagni, 2005).

[6] In this paper the translations are ours.

[7] Leibnitian letters to Wolf were published in *Acta Eruditorum Lipsiae*, Tom. V. ab an. 1711 ad an. 1719 Epist. G.G.L. ad V. claris. Ch. Wolfium. It is worth noting that Leibniz corresponded with most of the scholars in Europe: as a matter of fact he had over 600 correspondents (see for instance his *Commercium Phisosphicum et mathematicum* with Johann Bernoulli: Lebniz & Bernoulli, 1745).

1−1 = 0, it seem impossible to state that the final result can be ½" (Leibniz, 1716, p. 183).[8]

Leibniz noticed that if we "stop" the infinite series 1−1+1−1+... (so we consider a "series finita": Leibniz, 1716, p. 187), it is possible to obtain either 0 or 1 with the same "probability". As a matter of fact,

> "the *series finita* […] can have an even number of terms, and the final one is negative: 1−1, or 1−1+1−1, or 1−1+1−1+1−1 […] or it can have an odd number of terms, and the final one is positive: 1, or 1−1+1, or 1−1+1−1+1" (Leibniz, 1716, p. 187).[9]

Leibnitian original conclusion is the following:

> "When numbers' nature vanishes, our possibility to consider even numbers or odd numbers vanishes, too. […] So taking into account what is stated by the authors that wrote about evaluations, […] we ought to take the arithmetical average [of 0 and 1], i.e. the half of their sum; and in this case nature itself respects *justitiae* law" (Leibniz, 1716, p. 187).[10]

Hence the most probable value is the average of 0 and 1, that is ½.

It is worth highlighting this essential passage from finite cases (*series finita*) to the final infinite situation: as a matter of fact there is a clear and important change, and numbers' nature itself "vanishes". Would it be possible to relate this perspective to actual infinity? Of course we do not claim that Leibniz, by that, made explicitly reference to this conception. However his approach is interesting and may deserve further historical analysis.

Finally, Leibniz (1716, p. 188) conceded that "his argument was more metaphysical than mathematical, but went on to say that there was more metaphysical truth in mathematics than was generally recognized" (Kline, 1972, p. 446).[11]

2.2. Varignon and Riccati

Lagrange and Poisson also accepted previous argument; but Pierre de Varignon (1654-1722) noticed that in order to state:

$$\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-...+...$$

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[8] "Utrum 1−1+1−1+1−1+&c. in infinitum sit ½; & quomodo absurditas evitari possit, quae in tali enuntiatione se ostendere videtur. Nam cum infinities occurrere videatur 1−1 = 0, non apparet quomodo est veris nihilis infinities repetitis possit fieri ½" (Leibniz, 1716, p. 183).

[9] "*Series finita* […] vel enim constat ex *numero membrorum pari* & terminatur per − veluti: 1−1, aut 1−1+1−1, aut 1−1+1−1+1−1 […] vel *numero membrorum impari*, & terminatur per +, veluti: 1, aut 1−1+1, aut 1−1+1−1+1" (Leibniz, 1716, p. 187).

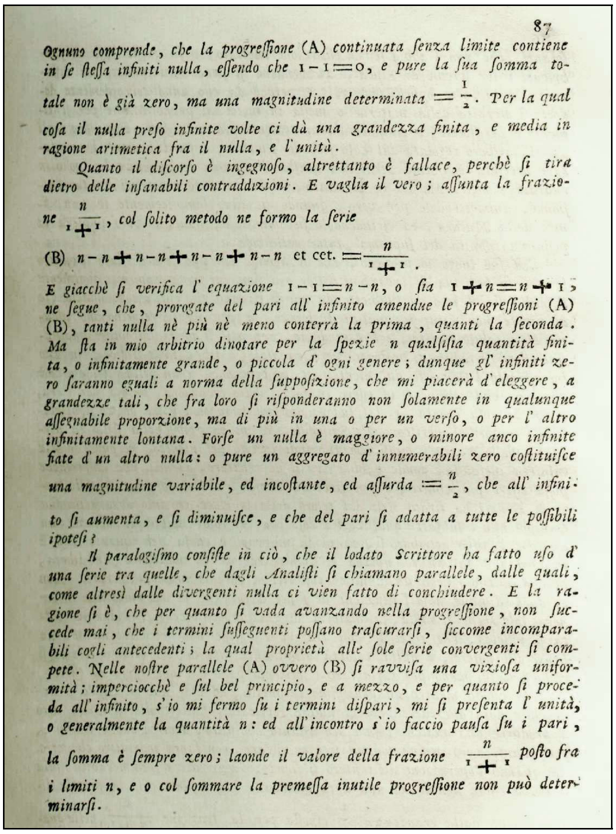
[10] "Tunc evanescente natura numeri, evanescit etiam paris aut imparis assignabilitas […]. Et quoniam ab iis qui de aestimatione scripsere, […] sumi debere medium Arithmeticum, quod est dimidium summae; itaque natura rerum eandem hic observat *justitiae legem*" (Leibniz, 1716, p. 187).

[11] "Porro hoc argumentandi genus, etsi Metaphysicum magis quam Mathematicum videatur, tamen firmum est: & aliorum Canonum *Verae Metaphysicae* (quae ultra vocabulorum nomenclaturas procedit) major est usus in Mathesi, in Analysi, in ipsa Geometria, quam vulgo putatur" (Leibniz, 1716, p. 188).

the condition $b<a$ is needed (Loria 1929-1933, p. 673), while the convergence of Grandi's series to ½ can be obtained by $a = b = 1$. So Varignon's note can be interpreted as a first consideration of the role of convergence.

Jacopo Riccati criticised the convergence of Grandi's series to ½; in *Saggio intorno al sistema dell'universo* (1754), he wrote:

> "[Grandi's] argument is interesting, but wrong because it causes contradictions. […] Let us consider $n/(1+1)$ and, by the common procedure, build $n-n+n-n$ etc. $= n/(1+1)$. If it is remembered that $1-1 = n-n$, or $1+n = n+1$, we have, in both series [in this series and in Grandi's], that there is the same quantity of zeroes" (Riccati, 1761, I, p. 87).[12]



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Ognuno comprende, che la progreſſione (A) continuata ſenza limite contiene in ſe ſteſſa infiniti nulla, eſſendo che $1-1=0$, *e pure la ſua ſomma totale non è già zero, ma una magnitudine determinata* $=\frac{1}{2}$. *Per la qual coſa il nulla preſo infinite volte ci dà una grandezza finita, e media in ragione aritmetica fra il nulla, e l'unità.*

Quanto il diſcorſo è ingegnoſo, altrettanto è fallace, perchè ſi tira dietro delle inſanabili contraddizioni. E vaglia il vero; aſſunta la frazione $\frac{n}{1+1}$, *col ſolito metodo ne formo la ſerie*

(B) $n-n+n-n+n-n+n-n$ et cet. $=\frac{n}{1+1}$.

E giacchè ſi verifica l'equazione $1-1=n-n$, *o ſia* $1+n=n+1$, *ne ſegue, che, prorogate del pari all'infinito amendue le progreſſioni* (A) (B), *tanti nulla nè più nè meno conterrà la prima, quanti la ſeconda. Ma ſta in mio arbitrio dinotare per la ſpezie n qualſiſia quantità finita, o infinitamente grande, o piccola d'ogni genere; dunque gl'infiniti zero ſaranno eguali a norma della ſuppoſizione, che mi piacerà d'eleggere, a grandezze tali, che fra loro ſi riſponderanno non ſolamente in qualunque aſſegnabile proporzione, ma di più in una o per un verſo, o per l'altro infinitamente lontana. Forſe un nulla è maggiore, o minore anco infinite fiate d'un altro nulla: o pure un aggregato d'innumerabili zero coſtituiſce una magnitudine variabile, ed incoſtante, ed aſſurda* $=\frac{n}{2}$, *che all'infinito ſi aumenta, e ſi diminuiſce, e che del pari ſi adatta a tutte le poſſibili ipoteſi?*

Il paralogiſmo conſiſte in ciò, che il lodato Scrittore ha fatto uſo d'una ſerie tra quelle, che dagli Analiſti ſi chiamano parallele, dalle quali, come altresì dalle divergenti nulla ci vien fatto di conchiudere. E la ragione ſi è, che per quanto ſi vada avanzando nella progreſſione, non ſuccede mai, che i termini ſuſſeguenti poſſano traſcurarſi, ſiccome incomparabili cogli antecedenti; la qual proprietà alle ſole ſerie convergenti ſi compete. Nelle noſtre parallele (A) ovvero (B) ſi ravviſa una vizioſa uniformità; imperciocchè e ſul bel principio, e a mezzo, e per quanto ſi proceda all'infinito, s'io mi fermo ſu i termini diſpari, mi ſi preſenta l'unità, o generalmente la quantità n: ed all'incontro s'io faccio pauſa ſu i pari, la ſomma è ſempre zero; laonde il valore della frazione $\frac{n}{1+1}$ *poſto fra i limiti n, e o col ſommare la premeſſa inutile progreſſione non può determinarſi.*

Riccati's argument deserves a brief remark; he writes $\frac{1}{2} = 1-1+1-1+\ldots$, "by the common procedure", then he introduces the infinite series: $n/2 = n-n+n-n+\ldots$ Let us compare the considered series; we can write:

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12 "Quanto il discorso è ingegnoso, altrettanto è fallace, perché si tira dietro delle insanabili contraddizioni. […] Assunta la frazione $n/(1+1)$, col solito metodo ne formo la serie $n-n+n-n$ etc. $= n/(1+1)$. E giacché si verifica l'equazione $1-1 = n-n$, o sia $1+n = n+1$, ne segue che prorogate del pari all'infinito amendue le progressioni […], tanti nulla né più né meno conterrà la prima quanti la seconda" (Riccati, 1761, I, p. 87).

$s = 1-1+1-1+1-1+... = (1-1)+(1-1)+(1-1)+... = 0+0+0+...$
$s' = n-n+n-n+n-n+... = (n-n)+(n-n)+(n-n)+... = 0+0+0+...$

Through this, Riccati concludes that Grandi's procedure is incorrect. Of course, nowadays, this argument cannot be accepted (it is based upon the "common procedure" referred to indeterminate series); however, Riccati's conclusion is correct:[13]

"The mistake is caused by the use of a series […] from which it is impossible to get any conclusion. In fact, […] it does not happen that the following terms can be neglected in comparison with preceding terms; this property is verified only for convergent series" (Riccati, 1761, I, p. 87).[14]

3. A BRIEF EXPERIMENTAL SURVEY

The educational use of historical references must be carefully controlled: as a matter of fact, the consideration of infinite series can cause inconsistencies in students' minds: for instance, if a pupil considers an infinite series as an arithmetical operation, the absence of the sum of Grandi's series can cause many doubts (and the influence of the didactical contract can be highlighted: Sarrazy, 1995).

We shall briefly consider pupils' opinions regarding Grandi's series. A test (Bagni, 2005) was proposed to students of two third-year *Liceo Scientifico* classes, total 45 pupils (aged 16-17 years), and of two fourth-year *Liceo scientifico* class, 43 pupils (aged 17-18 years; total: 88 pupils), in Treviso (Italy). Their mathematical curricula were traditional: in all classes, at the moment of the test, pupils did not know infinite series; they knew the concept of infinite set.[15]

We ask our students to consider "1–1+1–1+..." (studied "in 1703" by "the mathematician Guido Grandi"), taking into account that "addends, infinitely many, are always +1 and –1" and to express their "opinion about it" (time: 10 minutes; no books or calculators allowed):

Answers	the result is 0	26	29%
	the result is 1	3	4%
	the result can be either 0 or 1	18	20%
	the result is ½	4	5%
	the result is infinite	2	2%
	the result does not exist	5	6%
	no answer	30	34%

[13] Riccati's statement can be related to ideas that mathematicians were going to point out in the 18th century; finally, in *Disquisitiones generales circa seriem infinitam*, Gauss considered the notion of convergence correctly.

[14] "Il paralogismo consiste in ciò, che il lodato Scrittore ha fatto uso d'una serie […] dalle quali, come altresì dalle divergenti, nulla ci vien fatto di conchiudere. E la ragione si è, che […] non succede mai, che i termini susseguenti possano trascurarsi, siccome incomparabili con gli antecedenti; la qual proprietà alle solo serie convergenti si compete" (Riccati, 1761, I, p. 87).

[15] The researcher was not the mathematics teacher of the pupils, however, he was present in the classroom with the teacher and the pupils; the experience took place during a lesson in the classroom.

It is worth highlighting that the greater part of the pupils interpreted this question as an implicit request to calculate the "sum" of the considered infinite series. Only 5 students (6%) stated that it is impossible to calculate the sum of Grandi's series (and their answers were not provided with clear justifications); it should be remembered that 18 pupils, 20%, gave two "results"; 35 pupils (40%) gave a "result" (a finite or an infinite one) and many pupils gave no answer (34%).

Several students justified their answers in some interviews. Concerning pupils that stated that the sum of the infinite series 1–1+1–1+... is 0, some of them made reference to an argument by Grandi, quoted by Riccati, too ("If I always want to add 1 and –1, I can write (1–1)+(1–1) and so I can couple 1 and –1: so I am going to add infinitely many 0: I obtain 0": Marco, third year, and 15 other pupils). Students that stated that the sum of the considered series is ½ made reference to justifications similar to the argument by Leibniz-Wolf (for instance: "If I add the numbers I have 1, 0, 1, 0 and always 1 and 0. The average is ½": Mirko, fourth year).

Audio-recorded material and transcriptions allowed us to point out a salient short passage (1 minute and 35 seconds, 9 utterances):

[1] Researcher: "Why did you write that the result is ½?"[16]

[2] Mirko: "Oh, well, I start with 1, so I have 0, then 1, 0 and so on. There are infinitely many +1 and –1."[17]

[3] Researcher: "That's true, but how can you say ½?"[18]

[4] Mirko: "If I add the numbers, I obtain 1, 0, 1, 0 and always 1 and 0. The average is ½."[19]

[5] Researcher: "And so?"[20]

[6] Mirko: "The numbers that I find are 1, 0, and 1, 0, and 1, 0 and so on: clearly, every two numbers, one of them is 0 and one of them is 1. So these possibilities are equivalent and their average is ½."[21]

[7] Mirko: [after 12 seconds] "Perhaps my answer is strange, or wrong, but I don't see a different correct result: surely both the results 0 and 1 are wrong. If I say that the result is one of that numbers, for instance 1, I forget all the other numbers, an infinite sequence of 0."[22]

[8] Researcher: "So in your opinion both 0 and 1 cannot be considered the correct answer."[23]

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[16] "Perché hai scritto che il risultato è ½?"

[17] "Beh, all'inizio c'è 1, poi fa 0, poi 1, o e via. Ci sono infiniti +1 and –1."

[18] "Vero, ma perché ½?"

[19] "Se faccio le somme ottengo 1, 0, 1, 0 e sempre 1 e 0. La media è ½."

[20] "E allora?"

[21] "I numeri che si trovano sono 1, 0, e 1, 0, e 1, 0, sempre così: è ovvio, ogni due numeri uno è uno 0 e l'altro è un 1. C'è la stessa possibilità e la media fa ½."

[22] "Magari il mio è un discorso strano, magari anche sbagliato, ma non riesco a fare una cosa diversa: 0 e 1 non vanno bene di sicuro. Se dico che il risultato è uno di quelli, tipo 1, non conto tutti gli altri numeri, tutta la infinita fila di 0."

[23] "Dunque tu dici che 0 e 1 non sono risultati giusti."

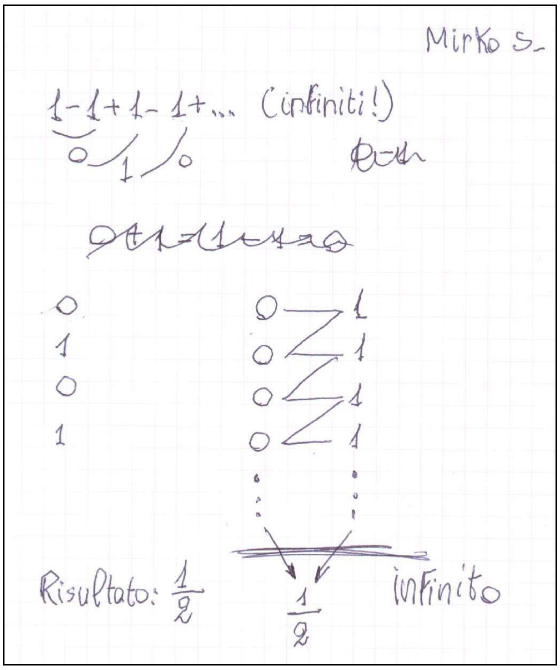
[9] Mirko: "Alright, and in this case what is the result? I wrote that ½ is the results of the operation because ½ is the average, so it is a number that, in a certain sense, contains both 0 and 1."[24]

Mirko stated that "every two numbers, one of them is 0 and one of them is 1" ([4]) and "the average (…) is a number that, in a certain sense, contains both 0 and 1"([9]). So he did not make explicit reference to the probability: he mainly tried to find a result for the considered problem, and this is an educational issue (influenced by the didactical contract); in the 18th century, the probabilistic argument was based upon a slightly different remark, according to which if we "stop" the infinite series 1−1+1−1+..., it is possible to obtain both 0 and 1 with the same "probability".[25]

So really students' justifications are sometimes similar to some examples from the history of mathematics (Furinghetti & Radford, 2002); but our present cultural context is quite different from the context that characterised the historical reference.

4. MIRKO'S PROTOCOL

With regard to the notion of infinity (potential infinity and actual infinity), it is interesting to examine Mirko's protocol.



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24 "Va bé e allora qual è il risultato? Io lì ho messo ½ come risultato dell'operazione perché ½ è la media, cioè quel numero che in un certo senso contiene 0 e 1."

25 Let us remember the importance of the probability in the mathematical researches in the 18th century (for instance, in *Acta Eruditorum 1682-1716*, we find either the quoted Leibnitian letter to Wolf or Bernoulli's *Specimina Artis Conjectandi*).

Mirko's use of visual elements is interesting: in particular, it is worth noting the horizontal sign by which Mirko divides the "finite" from the "infinite". After that sign, he writes "infinity" ("infinito").

This protocol reveals a clear difference, in Mirko's mind, between those situations: when we consider the first group of steps (referred to the *series finita*, so to speak), we have an alternating sequence of numbers, both 0 and 1 (and this sequence is highlighted by segments connecting the numbers). But the final situation ("infinity") is completely different: now we have no place enough for "two" numbers (0, 1), so we must write only one value after the arrows, i.e. the result: their average, ½.

Of course Mirko's protocol does not evoke a potential conception of infinity: one would think his visualization suggests that infinity is not just a continuous process that can be indefinitely lengthened. More precisely, infinity is referred to a single "entity", a single "place" after the arrows and the marked line. Nevertheless this situation is not enough to consider Mirko's approach with reference to actual infinity, but we can point out an ideal and interesting analogy with the aforementioned ancient Leibnitian solution.

However the main point to be discussed is the following: can we state that *historical* aspect, in particular Leibniz-Wolf's probabilistic argument, is really essential in order to suggest a conception grounded on actual infinity?

More properly, in our opinion, the crucial aspect is *educational*: the arrows, in Mirko's protocol, lead to the final "result of the operation", whose primary importance is emphasized by the didactical contract: the role of Leibniz-Wolf's probabilistic argument is minor, in this step. So we can state that considered elements ought to be connected in the following order:

Guido Grandi's series is *a problem to be solved*

(and)

A problem is solved if and only if
its *correct result* is provided [*didactical contract*]

(so)

Mirko has to pass from the two "partial results" 0 and 1
to *the* correct result (one and only one [*didactical contract*]) → *actual infinity?*

(so)

Being 0 and 1 seemingly equivalent,
Mirko calculates their average
("a number that, in a certain sense, contains both 0 and 1"):
by that, he respects "*justitiae* law"
(Leibniz, 1716, p. 187)

So the role of aforementioned historical references is really interesting, but it is strictly connected to educational aspects: and this leads us to state that Mirko's and Leibniz-Wolf's arguments are hardly referred to the "same" epistemological obstacle.

5. CONCLUDING REMARKS

In our opinion some examples from the history do help with the introduction of an important topic of the mathematical curriculum of High School.[26] Nevertheless, historical examples clearly stimulated many pupils, but an explicit institutionalisation by the teacher is clearly necessary. In order to conclude our reflections, we turn back to the theoretical framework mentioned at the beginning of this paper.

Is it possible to state, and to use in educational practice, a paralleling of history with learning processes? According to several researchers (e.g. Sfard, 1991, p. 10), the historical development of a concept can be regarded as the sequence of steps. In the early step the focus is mainly operational; the structural point of view is not a primary one: for example, as previously noticed, concerning infinite series, in this early step main questions of convergence were *not* considered (for instance, let us remember once again Riccati's aforementioned argument). A similar situation can be pointed out from the cognitive point of view: of course, in the early step pupils approach concepts mainly by intuition, without a full comprehension of the matter; then the learning becomes better and better.

Some experimental results seem to suggest that in the educational passage from the early step to the mature one we can see, in pupils' minds, reactions and doubts that we can find in the passage from the early step to the mature one as regards the *savoir savant* (Tall & Vinner, 1981).[27] But several issues ought to be considered: for instance, what do we mean by "pupils' minds"? More generally, can we still consider our mind as a "mirror of nature" (Rorty, 1979) and make reference to our "inner representations" uncritically? According to W.V.O. Quine,

> "epistemology, or something like it, simply falls into place as a chapter of psychology and hence of natural science. It studies a natural phenomenon, viz., a particular human subject" (Quine, 1969, p. 82).

Moreover R. Rorty underlines the crucial importance of "the community as source of epistemic authority" (Rorty, 1979, p. 380), and states:

> "We need to turn outward rather then inward, toward the social context of justification rather than to the relations between inner representations" (Rorty, 1979, p. 424).

Of course this theoretical perspective needs further research in order to be effectively applied in mathematics education. Nevertheless, we can state that a sociological approach is very important, and points out some difficulties (Bagni & D'Amore, 2005) from the educational viewpoint: since an *operational* conception can be considered before a *structural* one, as far as infinite series is concerned the passage from an

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[26] Of course in this research the considered sample is rather small; it would be necessary to identify sampling criteria and pre-course intuitions: so we cannot give general results.

[27] Since processes of teaching-learning take place nowadays, the *transposition didactique* can also be based upon the results achieved in the mature step of the development of the mathematical knowledge; with regard to teachers, of course, a relevant historical-phislosophical skill is needed.

operational conception to a structural one has been arduous, because of the necessity of some basic notions, like the limit concept,[28] which was not considered in many particular cultural contexts. In our opinion a crude paralleling of history with learning processes would connect two cultures referring to quite different contexts (Radford, 1997), so it cannot be used without a consideration of the social and cultural backgrounds.[29]

We can conclude that the introduction of infinite series in the classroom is not simple and several aspects can be considered.[30] The examined example is meaningful: we cannot forget that Mirko made reference to Leibniz-Wolf's probabilistic argument, and his behaviour can be considered in the sense of a first approach to actual infinity; but it is important to highlight that his choice is essentially related to the clause of didactical contract that emphasizes the importance of the result of a given problem or operation.

The main problem of the passage from finite to infinite is a cultural one, and historical issues are important in order to approach it, although, undoubtedly, the historical approach can be considered together with other (educational) approaches.[31] So let us finally quote T. Heiede, who states:

> "The history of mathematics is not just a box of paints with which one can make the picture of mathematics more colourful, to catch interest of students at their different levels of education; it is a part of the picture itself. If it is such an important part that it will give a better understanding of what mathematics is all about, if it will widen horizons of learners, maybe not only their mathematical horizons (…) then it must be included in teaching" (Heiede, 1996, p. 241).

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[28] This has relevant consequences for education. For instance, some remarks in the historical part can suggest the following issue: is it reasonable to introduce convergence in schools without prior introduction of the limit notion, since this is what happened historically? (Bagni, 2005).

[29] Let us moreover notice that frequently a better use of the developed systematic structure of mathematics for teaching runs counter to the direct paralleling of history with learning processes.

[30] For instance, embodiment (Lakoff & Núñez, 2000) is one of the most important issues of research into mathematics education and it is relevant to investigate further connections between perceptions and symbols. However the fundamental work by Lakoff and Núñez is devoted to cognitive aspects: the crucial point is the passage from finite to infinite; and metaphorical reasoning, clearly very important from the educational point of view, must be controlled by the teacher in order to avoid dangerous misguided generalisations (Bagni, 2000b).

[31] Several questions are still open: for instance, the reading of primary sources can be an important tool (Fauvel & van Maanen, 2000), but it needs a clear consideration of the historical evolution of representative registers (Bagni, forthcoming-a).

REFERENCES

Bachelard, G. (1938), *La formation de l'esprit scientifique*, VRIN, Paris.

Bagni, G.T. (2000a), Difficulties with series in history and in the classroom, Fauvel, J. & van Maanen, J. (Eds.), *History in Mathematics Education. The ICMI Study*, Kluwer Academic Publishers, 82-86.

Bagni, G.T. (2000b), "Simple" rules and general rules in some high school students' mistakes, *Journal für Mathematik Didaktik*, 21, 2, 124-138.

Bagni, G.T. (2005), Infinite series from history to mathematics education, *International Journal for Mathematics Teaching and Learning*, ISSN 1473 – 0111, http://www.ex.ac.uk/cimt/ijmtl/bagni.pdf.

Bagni, G.T. (forthcoming-a), Historical roots of limit notion. Development of its representative registers and cognitive development, *Canadian Journal of Science, Mathematics and Technology Education*.

Bagni, G.T. (forthcoming-b), Some cognitive difficulties related to the representations of two major concepts of set theory, *Educational Studies in Mathematics*.

Bagni, G.T. & D'Amore, B. (2005), Epistemologia, sociologia, semiotica: la prospettiva socio-culturale, *La matematica e la sua didattica*, 1, 73-89.

Barbin, E. (1994), Sur la conception des savoirs géométriques dans les Éléments de Géométrie, Gagatsis, A. (Ed.), Histoire et enseignement des Mathématiques: *Cahiers de didactique des Mathématiques*, 14-15, 135-158.

Brousseau, G. (1983), Les obstacles épistémologiques et les problèmes in mathématiques, *Reserches en Didactique des Mathématiques*, 4, 2, 165-198.

Cantoral, R. & Farfán, R. (2004), *Desarrollo conceptual del cálculo*, Thomson, Mexico.

Chevallard, Y. (1985), *La transposition didactique, du savoir savant au savoir enseigné*, La Penseé Sauvage, Grenoble.

Dauben, J.W. & Scriba, C.J. (2002), *Writing the history of mathematics: its historical development*, Birkhäuser, Basel, Switzerland.

Duval, R. (1995), *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*, Peter Lang, Paris.

Edwards, C.H. Jr. (1994), *The Historical Development of the Calculus*, Springer, Berlin.

Fauvel, J. & van Maanen, J. (Eds.) (2000), *History in Mathematics Education. The ICMI Study*, Dodrecht, Kluwer.

Furinghetti, F. & Radford, L. (2002), Historical conceptual developments and the teaching of mathematics: from philogenesis and ontogenesis theory to classroom practice, English, L. (Ed.), *Handbook of International Research in Mathematics Education*, 631-654, Lawrence Erlbaum, Hillsdale, New Jersey.

Grugnetti, L. & Rogers, L. (2000), Philosophical, multicultural and interdisciplinary issues, Fauvel, J. & van Maanen, J. (Eds.), *History in Mathematics Education. The ICMI Study*, 39-62, Dordrecht, Kluwer.

Hairer, E. & Wanner, G. (1996), *Analysis by its history*, Springer-Verlag, New York.

Heiede, T. (1996), History of mathematics and the Teacher. In Calinger, R. (Ed.), *Vita Mathematica*. The Mathematical Association of America, 231-243.

Kline, M. (1972), *Mathematical thought from ancient to modern times*, Oxford Un. Press, New York.

Lakoff, G. & Núñez, R. (2000), *Where mathematics come from? How the embodied mind brings mathematics into being*, Basic Books, New York.

Leibniz, G.W. (1716), Epistola G.G.L. ad V. Clariss. Christianum Wolfium, Professorem Matheseos Halensem, circa scientiam infiniti. *Excerpta ex Actis Eruditorum Lipsiensibus*, t. V suppl., 183-188.

Leibniz, G.W. & Bernoulli, Jo. (1745), *Commercium philosophicum et mathematicum*, Tomus primus. ab anno 1694 ad annum 1699. Tomus secundus, ab anno 1700 ad annum 1716. Lausanne-Généve, Bousquet.

Lizcano, E. (1993), *Imaginario colectivo y creación matemática*, Gedisa, Barcelona.

Loria, G. (1929-1933), *Storia delle matematiche dall'alba delle civiltà al tramonto del secolo XIX*, Sten, Torino, Italy (reprint: Cisalpino-Goliardica, Milano 1982).

Michieli, A.A. (1943), Una famiglia di matematici e di poligrafi trivigiani: i Riccati. I. Iacopo Riccati, *Atti del Reale Istituto Veneto di scienze, lettere ed arti*, CII,. II.

Piaget, J. & Garcia, R. (1983), *Psychogenèse et histoire des sciences*, Flammarion, Paris.

Quine, W.V.O. (1969), *Ontological Relativity and Other Essays*, Columbia University Press, New York.
Radford, L., Boero, P. & Vasco, C. (2000), Epistemological assumptions framing interpretations of students understanding of mathematics, Fauvel, J. & van Maanen, J. (Eds.), *History in Mathematics Education. The ICMI Study*, 162-167, Kluwer, Dordrecht.
Radford, L. (1997), On psychology, historical epistemology and the teaching of mathematics: towards a socio-cultural history of mathematics, *For the Learning of Mathematics*, 17(1), 26-33.
Rorty, R. (1979), *Philosophy and the Mirror of Nature*. Princeton University Press, Princeton NJ (page numbers are referred to: *La filosofia e lo specchio della natura*. Bompiani, Milano 2004).
Riccati, J. (1761), *Opere*, I, Giusti, Lucca.
Sarrazy, B. (1995), Le contrat didactique, *Revue Française de Pédagogie*, 112, 85-118.
Sfard, A. (1991), On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coins, *Educational Studies in Mathematics*, 22, 1-36.
Tall, D. & Vinner, S. (1981), Concept image and concept definition in Mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-169.
Wittgenstein, L. (1956), *Bemerkungen über die Grundlagen der Mathematik*. Basil Blackwell, Oxford.

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