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Exhaustion argument and limit concept in the History of Mathematics: educational reflections

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Abstract. In this paper we consider some epistemological issues related with the historical presentation of a mathematical concept, with particular reference to the educational introduction of infinitesimal methods. We present some theoretical frameworks and we underline the importance of non-mathematical elements. We conclude that the educational introduction of infinitesimal methods by historical references requires a correct socio-cultural contextualisation; nevertheless, historiographically, often an aprioristic platonic epistemological perspective is implicitly assumed.

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Giorgio T. Bagni

1. History and Didactics: a theoretical preface

Since the end of 19th century the History has been used into Didactics: F. Cajori's book *A History of elementary Mathematics with hints on methods of teaching* was published in 1896 (Rogers, 1995). Recent researches show that, in the educational field, frequently the first contact with a mathematical notion takes place in operative steps (Sfard, 1991) and this seems to support a parallelism between historical development and cognitive growth (Piaget, Garcia, 1983).

Important theoretical frameworks can be mentioned in order to link learning processes with historical issues. According to the epistemological obstacles perspective, particular systems of constraints in the History (*situations fondamentales*) must be studied for understanding existing knowledge (Radford, Boero, Vasco 2000). Obstacles are subdivided into epistemological, ontogenetic, didactic, cultural (Brousseau, 1989), so knowledge is considered isolately from other spheres; an important epistemological assumption is connected to the reappearance in teaching-learning processes of the same obstacles encountered by mathematicians in the History; the isolated approach of the pupil to the knowledge, without social interactions with other pupils and with the teacher, is moreover remarkable.

However can we directly compare different historical periods? What is the role played by socio-cultural factors that influenced the development of mathematical thought? It is impossible to interpret nowadays historical events without the influence of modern conceptions (Gadamer, 1975; Furinghetti, Radford, 2002); so we must accept our point of view and take into account that, when we look at the past, we connect two cultures that are "different [but] they are not incommensurable" (Radford, Boero, Vasco, 2000, 165).

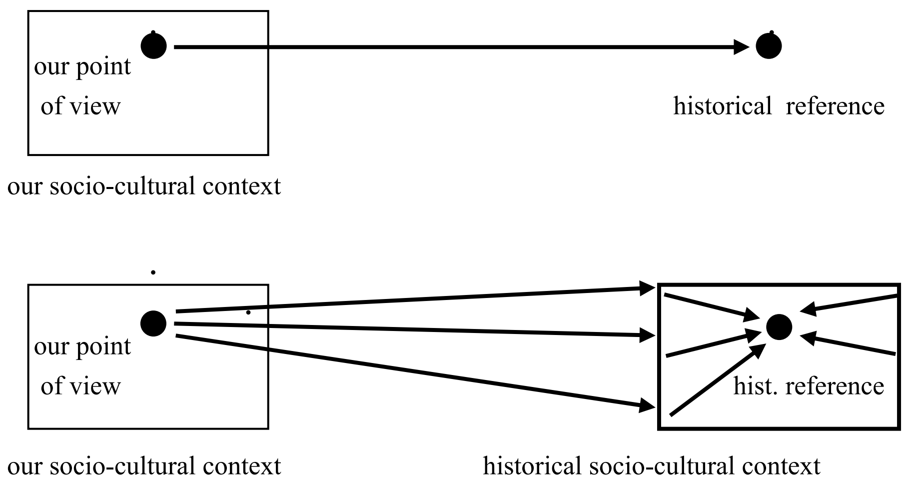

Table 1

According to the socio-cultural perspective by L. Radford, knowledge is linked to activities of individuals and is related to cultural institutions; it is built into a social context and the educational role played by historical elements must be considered with reference to different socio-cultural situations (concerning the role of signs: Radford, 2003):

> "Knowledge is a process whose product is obtained through negotiations of meaning which results in the social activity of the individuals and is encompassed by the cultural framework in which individuals are embedded" (Radford, 1997, 31).

In our opinion, knowledge can hardly be considered according to a classical teleological vision: let us explain this by a first example. In his *Quadratura parabolae*, Archimedes of Syracuse (287-212 b.C.) proved the following Proposition 23 (in *Elements*, IX-35, Euclid expressed a similar result):

> "If some quantities are such that everyone of them is four times the following one, all these quantities plus the third part of the lowest are 4/3 of the greatest one" (Frajese, 1974, 511; in this paper translations are ours).

Many centuries later, F. Viète (1540-1603) calculated the sum of a geometric infinite series; if we consider only the first quantity 1, being the remainder the third part of the last term, we have:

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots=\frac{1}{3}$$

so the sum of the (infinitely many) addends is:

$$1+\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots\right)=1+\frac{1}{3}=\frac{4}{3}$$

In 1655 A. Tacquet (1612-1660) published a similar result in *Arithmeticae theoria et praxis accurate demonstrata* (see moreover Wallis' *Arithmetica infinitorum*) and stated:

> "It is amazing that [ancient] mathematicians, who knew the theorem concerning finite progressions, did not consider the result concerning infinite ones, that can be immediately deduced by such theorem" (quoted in: Loria, 1929-1933, 517).

Tacquet made reference to ancient Mathematics without considering any historical contextualisation; although Aristotle (384-322 b.C.) implicitly underlined that the sum of a great number of addends (an infinite series, potentially considered) can be finite, Greek conceptions strictly distinguished actual and potential infinity; mathematical infinity, following Aristotle himself, is accepted only in potential sense: so it is meaningless to suppose any explicit consideration of infinite series. Tacquet's position, too, must be contextualised: we cannot suppose the presence of our epistemological awareness in 17th century.

2. Focus and methodology of our historical survey

When we introduce historically a mathematical concept, the selection of historical data is epistemologically relevant:

> "Data is never interesting in itself. Historical data will always be interesting with regards to the conceptual framework upon which the research program relies" (Radford, 1997, 28).

Problems are connected with their interpretation, which is always based upon cultural institutions and beliefs. Often original data are approached by later editions, so we must take into account the influence of editors' conceptions; E. Barbin writes:

“Every reading implies a re-interpretation and every writing implies a re-appropriation of ideas, of knowledge. Re-interpretations and re-appropriations of geometric knowledge by means of basic geometric works are referred (...) to epistemological conceptions. And these conceptions themselves must be considered in their historical contexts” (Barbin, 1994, 157).

We shall not propose a complete survey of the historical roots of the Calculus; we shall present some references in order to show that: (a) the educational introduction of infinitesimal methods by historical references requires a correct contextualisation; (b) historiographically, often an aprioristic platonic epistemological perspective is implicitly assumed (in the following table we summarise some quotations that we are going to consider in our work).

<table>
<tr><th>Author</th><th>Date</th><th>Platonic perspective</th><th>Socio-cultural contextualization</th></tr>
<tr><td colspan="4">The exhaustion argument</td></tr>
<tr><td>Euclid</td><td>300 b.C.</td><td>A first example of infinitesimal method</td><td>True knowledge cannot be reached through sense: the indirect method assured rigour to proofs</td></tr>
<tr><td colspan="4">Towards the infinitesimal</td></tr>
<tr><td>B. Cavalieri</td><td>1598-1647</td><td>A relevant step towards infinitesimal methods</td><td>17^{th}-century mathematicians needed effective tools</td></tr>
<tr><td>J. Wallis</td><td>1616-1703</td><td>His definition of limit “contains the right idea, but his wording is loose”</td><td>Rigour must be referred to the cultural institutions of the considered period</td></tr>
<tr><td colspan="4">“Vanishing quantities”?</td></tr>
<tr><td>G.W. Leibniz</td><td>1646-1716</td><td>Very success of his algorithms, but “uncertainty over concepts”</td><td>His position with reference to “vanishing quantities” is complex and changed in the time</td></tr>
<tr><td>L. Euler</td><td>1707-1783</td><td>His results are very important, but he didn’t consider difficulties with actual infinitesimals</td><td>His position is influenced by the features of the scientific frame of mind in the 17^{th} century</td></tr>
<tr><td colspan="4">Against “vanishing quantities”</td></tr>
<tr><td>J. d’Alembert</td><td>1717-1783</td><td>He refused vanishing quantities, but “his definition of limit lacked clear-cut phraseology”</td><td rowspan="2">Concerning the foundations of the Calculus, their ideas must be considered against the background of the Enlightenment</td></tr>
<tr><td>J.L. Lagrange</td><td>1736-1813</td><td>His attempt to reduce Calculus to Algebra “reveals his folly”</td></tr>
<tr><td colspan="4">The definition of limit</td></tr>
<tr><td>A.L. Cauchy</td><td>1789-1857</td><td>His definitions are still expressed in the verbal register</td><td>His definitions are expressed in the register available at his time</td></tr>
</table>

Table 2

3. Euclid and the exhaustion argument

The exhaustion argument is attributed to Eudoxus of Cnidus (405-355 b.C.; see for instance the Proposition XII-10: any cone is a third part of the cylinder with the same base and equal height): proofs by exhaustion argument are considered important infinitesimal processes and sometimes are proposed in classroom practice.

Proofs by exhaustion argument are based upon the following proposition, presented according to the version of *Elements* edited by F. Commandino (1509-1575):

Liber X, Propositio I. Duabus magnitudinibus inequalibus expositis, si à maiori auferatur maius, quàm dimidium,, ab eo, quod eliquum est rursus auferatur maius, quam dimidium;, hic semper fiat: relinquetur tandem quaedam magnitudo, quae minori magnitudine exposita minor erit (Commandino, 1619, 123r; see: Tartaglia, 1569, 177v, in Italian).

Proposition X-1. Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proved even if the parts subtracted are halves.

Euclid applied the so-called Eudoxus' postulate (which in *Elements* is a *definition*: in III-16 Euclid considered the set of rectilinear and curvilinear angles that is not a class of Archimedean magnitudes; so Greeks were not unaware of quantities that can be infinitesimal):

Liber V, Definitio IV. Proportionem habere inter se magnitudines dicuntur, quae multiplicatae se invicem superate possunt (Commandino, 1619, 57v; Tartaglia, 1569, 83r, in Italian).

Definition V-4. Magnitudes are said to *have a ratio* to one another which can, when multiplied, exceed one another.

Concerning the Proposition X-1, A. Frajese remembers the following fragment by Anaxagoras of Clazomenae (500?-428 b.C.):

"For neither is there a least of what is small, but there is always a less. For being isn't non-being" (Frajese, Maccioni, 1970, 596).

Can we refer such fragment to limit notion? Underlying the concept of limit there is the concept of the number system, so it would be necessary to consider the difference between magnitude and number in Greek contribution: the concept of number line is different as seen by Greeks or, for instance, by Cauchy (Tall, 1982). So a direct comparison between Anaxagoras and Cauchy is meaningless.

Let us now consider once again X-1: can we suppose the presence of a limit (in modern sense) in the exhaustion argument? M. Kline writes:

"The term *exhaustion* comes from the fact that the successive inscribed polygons *exhaust* the circle. (...) The term, as well as this loose description may suggest that the method is just approximate and just a step in the direction of rigorous limit concept. But (...) the method is rigorous. There is no explicit limiting process in it; it rests on the indirect method of proof and in this way avoids the use of a limit" (Kline, 1972, 83).

The non-equivalence is not only in the formal sense: most important differences pertain to the ontological realm (Radford, 2003). In the exhaustion argument we can recognise nowadays an infinitesimal process; but this interpretation is ours, so it is based upon modern conceptions: as Kline notices, "the indirect method of proof avoids the use of a limit". Euclid often applied the Proposition X-1, but he neither gave a definition of infinitesimal, nor proposed any particular denomination for infinitesimal processes. Greek beliefs and cultural institutions played a relevant role. As a matter of fact, Greek way of argumentation is shaped by the social and political context and was developed in the philosophical circles since the 5th century b.C. (Radford, 1997): such context cannot be forgotten when we interpret Greek contribution.

Let us moreover consider the following propositions:

Liber XII, Propositio I. Symilia polygona, quae in circulis describuntur, inter se sunt, ut diametrorum quadrata (Commandino, 1619, 211r; Tartaglia, 1569, 260r, in Italian).	**Proposition XII-1.** Similar polygons inscribed in circles are to one another as the squares on their diameters.
Liber XII, Propositio II. Circuli inter se sunt ut diametrorum quadrata (Commandino, 1619, 211v; Tartaglia, 1569, 260v, in Italian).	**Proposition XII-2.** Circles are to one another as the squares on their diameters.

Twenty centuries later, G. Saccheri (1667-1733) wrote:

"Euclid previously proved (XII-1) that similar polygons inscribed in circles are to one another as the squares on their diameters; by that it would be possible to deduce XII-2, by considering circles as polygons with infinitely many sides" (Saccheri, 1904, 104).

Saccheri's remark is interesting, in 17th century, but Greek mathematicians *never* used infinity according to this idea: Euclid's proof of Proposition XII-2 is completely different. A. Frajese correctly underlines:

"Clearly Saccheri is very close, from the chronological point of view, to Calculus foundation!" (Frajese, Maccioni, 1970, 930-931).

Exhaustion argument cannot be considered as an effective research tool: results to be proved by *reductio ad absurdum* must be known by intuition or by heuristic techniques, refused by Greeks as real proofs: in the Eleatan-Platonic mode of knowing, true knowledge (*episteme*) of a proposition P cannot be reached through sense; exhaustion argument was primarily introduced in order to assure rigour to proofs, by showing that the negation of P "leads us to the side of the *non-Being*" (Radford, 2003).

4. Towards the infinitesimal: Cavalieri, Wallis

In a different socio-cultural context, infinitesimals were considered in a very different way. B. Cavalieri proposed a new method and a denomination (*indivisibles*) but he did not give a definition of indivisible (Lombardo Radice, 1989). Surely his work can be considered a step towards the awareness of infinitesimal concepts; but this judgment is once again based upon our modern conceptions. Cavalieri's method, sometimes used in classroom practice, deserves a careful historical introduction.

Cavalieri had no preference for indirect methods (*reductio ad absurdum* was used only in Proposition II-12 of *Geometria indivisibilibus continuorom*; and some years later, Cavalieri gave another direct proof of such result in his *Exercitationes geometricae sex*: Lombardo Radice, 1989, 256):

> "He claimed his method was just a pragmatic device to avoid the method of exhaustion" (Kline, 1972, 350).

B. Pascal (1623-1662) and I. Barrow (1630-1677) expressed doubts about the utility of exhaustion argument (Bourbaki, 1960); P. de Fermat (1601-1665) wrote:

> "It would be easy to present proofs based upon Archimedean methods; I underline it once and for all, in order to avoid repetitions" (Fermat, 1891-1922, I, 257).

Seventeenth-century mathematicians needed effective tools: Cavalieri's method would not appear completely rigorous:

> "Cavalieri's indivisibles were criticised by contemporaries, and Cavalieri attempted to answer them; but he had no rigorous justification" (Kline, 1972, 350).

Nevertheless rigour must be investigated in its own conceptual context, in order to avoid the imposition of modern conceptual frameworks to works based upon different ones: "despite criticism", Cavalieri's method "was intensively employed by many mathematicians" (Kline, 1972, 350). It is clearly impossible that mathematicians in the History could refuse a method because of its foundational weakness that will be pointed out only through a modern approach. We agree with L. Radford:

> "It seems implausible that (a) the past mathematicians could have had a somewhat opaque vision of our modern concepts and (b) that they could have been struggling, in their remote epoch, to bring their concepts as close as they could have to our modern ones" (Radford, 1997, 27).

In fact, frequently historical evaluation is referred to our modern point of view: for instance, about J. Wallis, Kline writes:

> "Wallis, in the *Arithmetica Infinitorum*, advanced the arithmetical concept of the limit of a function as a number approached by the function so that the difference between this number and the function could be made less than any assignable quantity and would vanish ultimately when the process was continued to infinity. His wording is loose but contains the right idea" (Kline, 1972, 388).

"His wording is loose": what do we mean by that? If we investigate Wallis' correctness against our contemporary standards we must conclude that his expression is not rigorous. But such investigation would be historically weak: obviously Wallis' wording would not be correct, *nowadays*; but Wallis *was* rigorous, in his own way.

5. Vanishing quantities: Leibniz, Euler

The title of the present section does not suggest a direct comparison between the giants of the Mathematics: for instance, I. Newton (1642-1727) and G.W. Leibniz were responsive to his own primary intuition, which in the case of Newton was physical and in the case of Leibniz algebraic (Smith, 1959, 613-618, 619-626).

Leibnizian position is complex: he noticed in 1695 that "a state of transition may be imagined, or one of evanescence" in which "it is passing into such a state that the different is less than any assignable quantity; also that in this state there will still remain some difference, some velocity, some angle, but in each case one that is infinitely small" (quoted in: Kline, 1972, 386); Kline writes:

"However, Leibniz also said, at other times, that he did not believe in magnitudes truly infinite or truly infinitesimal. (...) It is apparent that neither Newton nor Leibniz succeeded in making clear, let alone precise, the basic concepts of the Calculus: the derivative and the integral. Not being able to grasp these properly, they relied upon the coherence of the results and the fecondity of the method to push ahead without rigour" (Kline, 1972, 387).

C.B. Boyer writes:

"It is nevertheless clear that Leibniz allowed himself to be carried away by the very success of his algorithms and was not deterred by uncertainty over concepts" (Boyer, 1985, 442).

How can we state "uncertainty over concepts" in Leibnizian thought? We can recognise it just nowadays, by our mathematical and epistemological conceptions; we agree with F. Enriques, who notices:

"It is not clear in Leibnizian works if these increments must be interpreted only in potential way, like variable and evanescent quantities, or as actual infinitesimal" (Enriques, 1938, 60).

making reference, by that, to a problem residing into our modern interpretation of Leibnizian ideas.

L. Euler's ideas about infinitesimal quantities are interesting, although a parallelism between Leibniz and Euler cannot be stated uncritically. Euler in his *Institutiones calculi differentialis* (1755: Caput III, *De infinitis atque infinite parvis*) argued:

"Nullum autem est dubium, quin omnis quantitas eousque diminui queat, quoad penitus evanescat, atque in nihilum abeat. Sed quantitas infinite parva nil aliud est nisi quantitas evanescens, ideoque revera erit = 0. Consentit quoque ea infinite parvorum definitio, qua dicuntur omni quantitate assignabili minora: si enim quantitas tam fuerit parva, ut omni quantitate assignabili sit minor, ea certe non poterit non esse nulla; namque nisi esset = 0, quantitas assignari potest ipsi aequalis, quod est contra hypothesin" (Euler, 1787, I, 62-63).

"There is no doubt that every quantity can be diminished to such an extent that it vanishes completely and disappears. But an infinitely small quantity is nothing other than a vanishing quantity and therefore the thing itself equals 0. It is in harmony also with that definition of infinitely small things by which the things are said to be less than any assignable quantity; it certainly would have to be nothing; for unless it is equal to 0, an equal quantity can be assigned to it, which is contrary to the hypothesis" (Kline, 1972, 429).

Unfortunately Euler did not see the possibility that a vanishing quantity can be a different kind of quantity from a numerical constant; he was aware of problems with actual infinitesimals, but in mathematical practice he preferred a different approach (*Introductio in Analysin Infinitorum* Chapter I-7: Euler, 1796, 84-91). Connections between Mathematics and socio-cultural context are fundamental: Euler's approach is not just "tuned in" to applicative features of the scientific frame of mind in the 17th century (Crombie, 1995) and such situation requiresa a deep study.

6. Necessity of rigour: d'Alembert, Lagrange

J.B. d'Alembert was

"an unusual combination of caution and boldness" (Boyer, 1985, 492).

For instance, d'Alembert conceptions about Calculus deserves a careful interpretation; he refused Leibnizian and Eulerian assumptions about differentials and in 1767 stated that a quantity "is something or nothing" and "the supposition that there is an intermediate state between these two is a chimera" (quoted in: Boyer, 1985, 493). Of course such point must be considered first of all with reference to d'Alembert's rich personality, linking the Jansenist education with his friendship with Voltaire; moreover it must be seen against the background of the Enlightenment (Grimsley, 1963). Boyer concludes:

"D'Alembert denied the existence of the actually infinite, for he was thinking of geometrical magnitudes rather than of the theory of aggregates proposed a century later. D'Alembert's formulation of the limit concept lacked the clear-cut phraseology necessary to make it acceptable to his contemporaries" (Boyer, 1985, 493).

We agree: of course it was impossible for d'Alembert to perceive by intuition ideas introduced by Cantor. However, concerning the lack of clarity, the judgment needs a bit of caution. In the article on *Limit* written for the *Encyclopédie* he stated that one quantity is the limit of a second variable one if the second can approach the first quantity closer than by any assignable quantity, without coinciding with it. This statement is weak, and wrong if compared with the correct modern notion of limit, so that

"the imprecision in this definition was removed in the works of nineteenth-century mathematicians" (Boyer, 1985, 493).

But d'Alembert's position must be framed into a particular socio-cultural context, nearly seventy years before the publication (1821) of Cauchy's treatise!

A quotation can be devoted to J.L. Lagrange, who in 1797 (*Théorie des fonctions analytiques*: Lagrange, 1813) tried to reduce Calculus to Algebra; Kline writes:

"Lagrange made the most ambitious attempt to rebuild the foundations of the Calculus. The subtitle of his book reveals his folly. It reads: *Containing the principal theorems of the differential Calculus without the use of the infinitely small, or vanishing quantities, or limits and fluxions, and reduced to the art of algebraic analysis of finite quantities*" (Kline, 1972, 430).

Can we consider Lagrange's attempt just as a "folly"? He tried to overcome the weakness of the Calculus. Surely his idea is based upon wrong assumptions (it met great favour for some time, but later it was abandoned: Kline, 1972, 432): however such judgment needs our modern epistemological conceptions and our mathematical skill.

7. Final reflections

So let us consider A.L. Cauchy who in his *Cours d'Analyse algébrique* gave the following definitions:

"When values of a variable approach indefinitely a fixed value, as close as we want, this is the *limit* of all those values. For instance, an irrational number is the limit of the different fractions that gave approximate values of it (…). When values of a variable are (…) lower than any given number, this variable is an *infinitesimal* or an infinitesimal magnitude. The limit of such variable is zero" (Cauchy, 1821, 4).

Cauchy finally introduced the distinction between *constants* and *variable quantities*, although he had no formal axiomatic description of real numbers. It is educationally interesting to underline that Cauchy's verbal formulation was expressed in the paradigm available at the time: nowadays it can lead to the use of different representation registers.

As we underlined, presented examples are not a full collection of historical data referred to limit notion: a number of authors are still missing, e.g. L. Valerio (1552-1618), K.T.W. Weierstrass (1815-1897), A. Robinson (Robinson, 1966). For instance, Weierstrass' definition of limit allows a modern symbolic representation, although it would be misleading to make reference to a single symbolic register: there are different registers in different communities of practice. Leibniz, Newton, Cauchy had their own symbolic registers which differ from each other and differ, too, from that of Weierstrass (Bagni, forthcoming).

The passage from discrete to continuum is a cultural problem and historical issues are important in order to approach it. History gives us the possibility of a metacognitive reflection and the possibility to achieve a socio-cultural comprehension of historical periods (Furinghetti, Somaglia, 1997). These possibilities are linked: the transfer of some situations from History to Didactics needs a wider cultural dimension keeping into account non-mathematical elements, too (Radford, 1997). An "internalist" History, so a conception of the development of Mathematics as a pure subject, isolated from "external" influences, is hardly useful in education (Grugnetti, Rogers, 2000, 40).

We have considered mathematical texts; moreover we must keep in mind that

> "Mathematics is not just text; it lives in the minds of people and can, to an extent, be disclosed by interpreting the artefacts they have produced [and] these artefacts, inscriptions, instruments, books and technical devices, have been developed in particular places for particular reasons" (Grugnetti, Rogers, 2000, 46).

Further researches can be devoted to point out connections between socio-cultural contexts and mathematical ideas and artefacts in various historical periods, and to clarify educational possibilities (for instance, Radford, 2003, studies the links between the symbolic structure and the modes of knowing, both of them related to socio-historico-economic dimension).

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