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Visualization and Didactics of Mathematics in High School: an experimental research *

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Abstract. It is well-known that the role of semiotic representations is important in the learning of mathematics: in this paper the influence of visualization on some algebraic and analytic methods is studied by three tests in Italian High School (pupils aged 14-19 years). We analyze some didactical implications connected to visualization: for example, the graphic representation (in particular the Cartesian graph of a function) is often tacitally considered the main aim of the whole study of a function; this procedure may be uneffective for the correct characterization of concepts and for the development of the ability to use and to coordinate registers of representation.

PREFACE

> "People remember visual aspects of a concept better than its analytical aspects"
>
> S. Vinner (1992, p. 212)

By visualization, it is possible to represent "externally" mathematical objects. Some Authors, in the last few years, worked about several problems connected to the representation in mathematics. R. Duval notes that «mathematical objects are not directly accessible to the perception... as objects generally said 'real' or 'phisical'»; so «different semiotical representations of a mathematical object are absolutely necessary»; he underlines that «it is the object which is important, and not its different semiotical representations» and that «the distinction between an object and its representation is therefore a strategical point for the comprehension of mathematics» (Duval, 1993, pp. 37-38). The presence of different registers of representation is, in Duval's opinion, very important: «The cognitive functioning of human thought is inseparable from the existence of a variety of semiotic registers of representation. If we call *sémiosis* the learning of the production of a semiotical representation and *noésis* the conceptual learning of an object, we must affirm that *sémiosis* is inseparable from *noésis*» (Duval, 1993, pp. 39-40; Duval, 1995).

(*) Some results from this work were published (in Italian) in: Bagni, G.T. (1997), La visualizzazione nella didattica della matematica: *L'insegnamento della matematica e delle scienze integrate*, 20B, 4, 309-335.

Among all kinds of representation of a mathematical object, the graphic representation is very important in mathematical education. Let us remember once again Duval, who affirms that «graphic representations are semiotic representations as much as the geometrical figures, algebraic writing or the common language»; but learning by graphic representations «needs a particular work» and «it is impossible to rely their use on spontaneous interpretation of pictures and of images» (Duval, 1994b).

A well-known work by E. Fischbein is specifically devoted to visual representation of mathematical objects and to its great importance in didactics of mathematics (Fischbein, 1993): by his "theory of figural concepts", Fischbein states that «the integration of conceptual and figural properties in unitary mental structures, with the predominance of the conceptual constraints over the figural ones, is not a natural process. It should constitute a continuous, systematic and main preoccupation of the teacher» (Fischbein, 1993, p. 156). So if by "figural concept" we mean a «fusion between concept and figure» (Fischbein, 1993, p. 143), we can underline, in Fischbein's words, that «the processes of building figural concepts in student's mind should not be considered a spontaneous effect of usual geometry courses» (Fischbein, 1993, p. 156).

The visualization as a way to represent mathematical objects is therefore fundamental in didactics of mathematics (Vinner, 1992); in particular, its role is essential in some "chapters" of the mathematical education in High School. In this work we shall consider two important topics of the traditional curriculum of Italian *Liceo scientifico* (High School) and we shall examinate the status of the didactics about such topics in reference to the visualization.

Our aim is therefore to examinate the important role of the visualization in the mathematical education of High School, to show that an incorrect didactical practice can bring to its wrong and uneffective employment (sometimes, as we shall see, having a harmful effect); finally we shall suggest some directions for its possible revaluation.

We shall examinate a recent work by M. Kaldrimidou about visualization (1994-1995), and in particular about the possibility to visualize some basic algebraic techniques (Kaldrimidou, 1995). In a previous work, M. Kaldrimidou noted: «We noticed in the students the absence of a systematical reflexion about mathematics and about the ways to acquire it, together with some stereotyped conceptions... Mental images, questions about their mutual connections and... strategies utilized, ... deserve, in our opinion, to be deeper investigated» (Kaldrimidou, 1987, pp. 156-157).

We shall examinate moreover the role of the visualization in the teaching and the learning of function, one of the main topics of the traditional curriculum of High School; a function can be "described" in several ways: it can be visualized, for example, by its Cartesian graph and this technique has important didactical consequences.

Let us show our work by:

1. Visualization of some algebraic techniques

A research by
M. Kaldrimidou
(1994-1995)

Test 1
(Visualization in students' opinion)

2. Visualization of the concept of function

Test 2
(Visualization in the learning of the concept of function, 3rd class)

Test 3
(Visualization in the learning of the concept of function, 5th class)

3. Conclusion

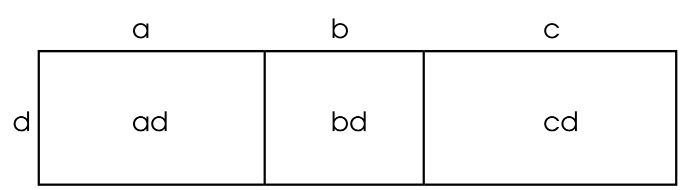
Visualization in traditional curriculum of High School:
status, possibilities, future.

1. ALGEBRA AND GEOMETRY

1.1. THE GREEK "GEOMETRIC ALGEBRA"

Some basic algebraic formulas can be usefully visualized by Greek "geometric algebra" (term by H.G. Zeuthen: Van der Waerden, 1983), introduced in II Book of Euclidean *Elements*; the main idea of this interesting technique is the representation of real numbers by some geometrical quantities (for example, by segments). So many operations can be visualized by figures: if two numbers are identified by two segments, their product corresponds to a rectangle having such segments for dimensions; so the equality of products is visualized by the equality of the areas of the corresponding rectangles. By that, it is possible to state general rules, related to real numbers.

Proposition 1 of II Book of *Elements*, for example, expresses gracefully the distributive property of multiplication in regard to addition.



Proposition 1 of II Book of *Elements*. If two segments are given, and one of them is divided in two (or more) parts, the area of the rectangle contained between the two segments is equal to the sum of the areas of the rectangles contained between the segment not divided and all the parts of the other (Euclid, 1970, p. 159).

The proofs of propositions of geometric algebra can be obtained directly by figures, so they are known by intuition by pupils and they are very useful from the didactic point of view (Dieudonné, 1989, p. 43).

1.2. THE STATUS OF VISUALIZATION

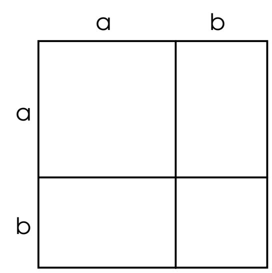
Images give to students the possibility to approach gradually the algebraic abstraction; but, as we shall see, visualization itself is often considered suspiciously by students and by teachers of mathematics. A work by M. Kaldrimidou underlines a negative, unfavourable consideration of the visual representations; a test was proposed to some students (College, 3rd year of the mathematical course) and to teachers of mathematics, in Greece (Kaldrimidou, 1995).

Let us produce one of the questions of the test by M. Kaldrimidou; this is referred to the 4th Proposition of Euclidean geometric algebra (that is: If a segment is divided, the area of the square of the whole segment is equal to the sum of the areas of the squares of the two parts and of the double of the area of the rectangle contained between such parts: Euclid, 1970, p. 163).

The question is the following:

“**1**. $(a+b)^2 =$
$= (a+b)(a+b) =$
$= a^2+ba+ab+b^2 =$
$= a^2+2ab+b^2$

2.



Between the two techniques to show that: $(a+b)^2 = a^2+2ab+b^2$, what is the most appropriate, from the mathematical point of view? Justify your answer”.

The main part of the students and of the teachers chose the analytical representation rather than the visual one:

Students		**Teachers**	
Analytical	68.3%	Analytical	61.5%
Visual	26.7%	Visual	23.1%
No choice	5.0%	No choice	15.4%

In the justifications given by students and by teachers choosing the analytical representation it is possible to find opinions revealing a negative consideration of visual representations. Let us resume the main doubts expressed:

The visual representation was sometimes considered a technique not suitable to represent informations, in mathematics; this depends upon the specificity of every visual representation;

Sometimes, teachers' justifications revealed epistemological obstacles («General methods are preferred, they try to have complete theories... Sometimes the algebraic kind of the method is enough to consider the solution more 'mathematical'. Visual representations do not have these features, so they are not considered equivalent to algebraic techniques»: Kaldrimidou, 1995);

For the students, visual representations were often cause of fear, of doubt: really the *didactical contract* seems to assign a large importance to algebraic expression to the detriment of the figural one (Brousseau, 1987).

1.3. METHOD AND RESULTS OF TEST 1

Previous results are referred to College students and to teachers, in Greece. We proposed once again the test to some Italian students to investigate the status of the visualization (of the algebraic techniques) in Italian High School.

The question remembered in the previous paragraph was proposed to 105 students of four classes of the last two courses (4th class and 5th class) of a *Liceo scientifico* (High School) in Treviso, Italy.

The main part of the students chose the analytical representation:

Analytical representation	62 students	60%
Visual representation	30 students	29%
No choice	13 students	11%

So these results are in agreement with the results obtained by Kaldrimidou (although differences are lower).

1.4. JUSTIFICATIONS GIVEN BY STUDENTS (TEST 1)

Several students gave justifications, subdivided in the following groups:

Some pupils (18 students, 17%) choosing the analytical representation underlined positively the presence of all the passages (this seems to make the work reliable and convincing); let us relate some justifications: "Algebraic method is better because it shows all the passages" (Luisa, 4th class). 'The algebraic method is more appropriate because there is the factorization" (Luca, 4th class). 'The proof of the analytical method is the correct development of a power; the other is just a figure and we cannot know if all the measures are correct or not" (Chiara, 4th class). It is evident the uncertainty about measuring of segments, the difference between the consideration of the real numbers a, b in algebraic field and as segments' measures.

Some pupils (6 students, 6%) choosing the analytical representation underlined that the best resolution for an algebraic problem is an algebraic one;

let us report some justifications: "The algebraic method is better: the proof of an algebraic expression is better by an algebraic method" (Marco F., 4th class). "The algebraic method is the most appropriate because it works upon an algebraic property by an algebraic technique" (Anna, 5th class). As we shall see later, in this interesting case, the obstacle of a representation different from the question is evident: "algebraic" and "geometrical" languages are equivalent, but the acknowledgement of such equivalence is an obstacle for the students.

Some pupils (4 students, 4%) choosing the analytical representation underlined the difficulty of the visual resolution, not usual: "I prefer to solve $(a+b)^2$ by algebraic method because I usually work with numbers more than with images" (Paola, 4th class). "The algebraic solution is better because it is surely correct and because it can be utilized also when it is difficult to visualize the operation" (Cristina, 5th class).

For pupils choosing the visual representation, justifications are based upon its evidence, its simplicity and the absence of calculations: "I prefer the visual method: the proof is evident, it needs no calculations" (Fabrizio, 4th class). "The algebraic meth od is not appropriate because it is only a mechanical proceeding based upon calculations" (Francesco, 5th class).

1.5. ANALYSIS OF JUSTIFICATIONS GIVEN BY STUDENTS AND CONCLUSIONS ABOUT TEST 1

Previous considerations, referred to a particular utilization of the visualization ([1]), are based just upon a check of students' preferences, so they cannot be considered as a deep examination of the status of didactics. However they show a rather clear situation: many students (17% on the total, and nearly the third part of the students choosing the analytical resolution) underlined the reassuring presence of all algebraic passages opposing to the unreliability, to the difficulty of the visual technique.

Many students are mainly worried about the choice of a sure and reliable solving method; none of them, for example, noticed that the mistake that identifies $(a+b)^2$ in a^2+b^2 (without the so-called "double product") is nearly impossible if the visual representation is correctly considered, while it is not infrequent if the resolution is just based upon the mnemonical, mechanical application of the formula $(a+b)^2 = a^2+2ab+b^2$.

The statement by which algebraic character of the problem should deserve an algebraic resolution is very interesting (6% on the total, and nearly the tenth part of the students choosing the analytical resolution): so several students are forced by a clause of the *didactical contract* to solve the exercise trying to meet teacher's approval. Of course, this behaviour limits (and probably worsens) the possibilities of the resolution of the problem.

([1]) About the basic role of image in didactical proceeding, see D'Amore, 1995. About limits, see: Gagatsis. & Dimarakis, 1996; Dimarakis & Gagatsis, 1997.

So we can conclude that the visualization of algebraic techniques is often considered suspiciously by students in High School. But it would be wrong to state that about *all* visual methods: as we shall see, for example, didactics of the functions is strongly based upon visual techniques: a connection that sometimes brings to identify the study of a function with the drawing of its Cartesian graph. From this point of view, the work by M. Kaldrimidou needs a closer inspection.

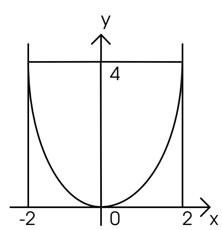
2. FUNCTIONS

2.1. FUNCTIONS AND CARTESIAN GRAPHS

In the didactics of the functions, the use of images to make the connection between elements of the domain and their corresponding values immediately evident is very frequent; for example, the visualization of the correspondence that associates to every real number its square can be obtained by the representation by arrows.

...	$\rightarrow$	...
-2	$\rightarrow$	4
$-\sqrt{3}$	$\rightarrow$	3
0	$\rightarrow$	0
1	$\rightarrow$	1
5/4	$\rightarrow$	25/16
6	$\rightarrow$	36
...	$\rightarrow$	...

The Cartesian graph is another graphic representation of a function; the relation previously examined can be visualized by the parabola denoted by the equation $y = x^2$.



Let us examinate in particular this last method of representation, very important in the traditional mathematical curriculum of High School (2).

First of all, let us note that usually (in several text-books) the equation $y = f(x)$ identifies the points whose Cartesian coordinates belong to the set:

$$\{(x;\, y) \in D \times \mathbf{R}:\, y = f(x)\} \qquad \text{(where } D \subseteq \mathbf{R} \text{ is the domain of the function } f\text{)}$$

(2) Of course, the representation of Cartesian graph depends upon several parameters: units, considered domain, range... Moreover, it is now changing strongly with the use of graphic calculators. Further researches would be devoted to these topics.

Sometimes, equation $y = f(x)$ is directly used to signify the function f, ("*let us consider the function* $y = f(x)$...*"), or its Cartesian graph ("*let us consider the curve* $y = f(x)$...*"). Of course, these are *misuses*: a *function* f $D \to \mathbf{R}$ is a particular *relation*, so a particular *subset of* $D \times \mathbf{R}$ (and it is neither an *equation*, nor a *curve*...); the equation $y = f(x)$ would be used to *represent* the function f. The Cartesian graph of $y = f(x)$ is just a *subset of the plane*, connected with $\{(x; y) \in D \times \mathbf{R}: y = f(x)\}$ by a "one-to-one" correspondence.

Moreover, the visualization is often connected immediately to *continuous* functions and the continuity is often related to graphical features of the function: a function is said *continuous* in a point of its domain when its Cartesian graph, in correspondence to such point, can be drawn... *without taking off the pencil from the sheet of paper*: in the College text-book by S.M. Nikolskij this "definition" is remembered (Nikolskij, 1985, p. 88). A continuous function in all its own domain is therefore a function whose graph can be drawn *continuously*, without "interruptions", without "tears" or "jumps", in all the fixed domain (T.M. Apostol too remembers "graphic irregularities" for functions that are *not* continuous: Apostol, 1969, I).

Of course, we do not want to deny the importance and the great utility of the visualization in didactics of the functions: but an uncontrolled and exaggerated use of visualization could bring to incorrect and harmful situations.

Not always, in High School, students can clearly distinguish the concept of function from its graphic visualization: for example, sometimes a function whose Cartesian graph is impossible to be drawn becomes a remarkable obstacle. Moreover, in exercises, students work with continuous functions (their graphs, for example, are straight lines, parabolas...) and this practice brings the students to consider the continuity as a common rule. In other words, pupils associate directly to the concept of *"function"* its graph, so a *"curve"* with its *"continuity"*, and they do not realize that a *continuous function* should be considered a (very) particular case of (general) function.

Some classical examples from Calculus can be useful to investigate the introduction of the concept of function, and in particular of continuous function (Van Rooj & Schikhof, 1982). A function whose examination is didactically important is *Dirichlet's function* ([3]), introduced, for every real number x, by following definition (Bagni, 1993, p. 468):

DEFINITION 1. Let Dirichlet's function $x \to f(x)$ be the real function that:

- if real x *is rational*, then $f(x) = 0$;
- if real x *is irrational*, then $f(x) = 1$.

([3]) Of course, the didactical importance of Dirichlet's function strongly depends on the aims of the curriculum.

Some Authors consider Dirichlet's function only in [0; 1]; of course, the didactical importance of this example does not change (Prodi, 1970, p. 308) ([4]).

It is possible to show that Dirichlet's function is *not* continuous for every real number x (the proof is easy; see: Bagni, 1994). It is important to underline that *the intuitive evaluation of the discontinuity of Dirichlet's function cannot be directly based upon the examination of its Cartesian graph*: the Cartesian graph of Dirichlet's function cannot be drawn (of course it can be drawn only for a *finite* set of points on the straight lines whose equations are $y = 0$, $y = 1$).

Moreover, we shall consider the function given, for every real number x, by following definition (Gelbaum, 1961, p. 124; Gelbaum & Olmsted, 1979, p. 34):

DEFINITION 2. Let Gelbaum's function $x \rightarrow f(x)$ be the function that:

- if real x *is rational*, $x = m/n$, being m an integer, n a positive integer, and m/n irreducible, then: $f(x) = 1/n$;
- if real x *is irrational*, then: $f(x) = 0$.

(Notice that it is a well-defined function: if x is rational, $x = m/n$, being m an integer, n a positive integer, such that m/n is irreducible, then m, n are univocally determined: Gelbaum, 1962, p. 53).

The study of Gelbaum's function cannot be based upon the examination of its graph: it is impossible to visualize its graph (not even approximately, like for Dirichlet's function). It is possible to show that Gelbaum's function is continuous for every x real *irrational* and it is not continuous for every x real *rational* ([5]): of course, this continuity for x irrational cannot be referred to the examination of its Cartesian graph.

2.2. METHOD AND RESULTS OF TEST 2

The following test was proposed to students belonging to three 3rd classes of a *Liceo scientifico* (High School) in Treviso, Italy, total 75 students (their mathematical curricula were traditional; they knew the concept of function and several possibilities to express a function; Cartesian graphs were introduced):

A) The following relations among real numbers are given; for everyone of them, draw (if it is possible) its Cartesian graph and say if it is a function:

1) R_1 is the relation such that (for every $x \in \mathbf{R}$) $R_1(x) = 2x$.

2) R_2 is the relation such that (for every $x \in \mathbf{R}$) $R_2(x) = 1$.

([4]) We underline that sometimes, according to a simple didactical introduction, a function must be defined by a *single* functional relation (like $f(x) = ...$); with this (personal) criterium, Dirichlet's function is not considered as a function.

([5]) The proof is not trivial: a summary of it is given in: Bagni, 1994, pp. 30-31.

3) R_3 is the relation such that:

- if real x *is rational*, then $R_3(x) = 0$;
- if real x *is irrational*, then $R_3(x) = 1$.

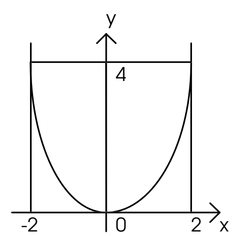
4) R_4 is the relation such that:

- if real x *is rational*, $x = m/n$, being m an integer, n a positive integer, and m/n irreducible, then: $R_4(x) = 1/n$;
- if real x *is irrational*, then $R_4(x) = 0$.

B) Among the following ways to express the relation that to every real number associates its square, what is the most appropriate, from a mathematical point of view?

1) $\mathbf{R}\rightarrow\mathbf{R}$
$x\rightarrow x^2$

2) $x\rightarrow y$



3) $...\rightarrow...$
$-2\rightarrow 4$
$-\sqrt{3}\rightarrow 3$
$0\rightarrow 0$
$1\rightarrow 1$
$5/4\rightarrow 25/16$
$6\rightarrow 36$
$...\rightarrow...$

Let us notice that point 2 of the question B of the test shows a curve... just *similar* to a parabola, being given *only* the points (–2; 4), (0; 0) and (2; 4); that representation is therefore incomplete, regard to the relation that to *every* real number associates its square (in spite of that, as we shall see, several pupils chose this way to represent the given relation). The results were the following:

Question A:

	Graph correctly drawn	It is a function	It is *not* a function	No answer
A1	69 (92%)	71 (95%)	1 (1%)	3 (4%)
A2	61 (81%)	54 (72%)	19 (25%)	2 (3%)
A3	0 (0%)	34 (46%)	31 (41%)	10 (13%)
A4	0 (0%)	21 (28%)	40 (53%)	14 (19%)

Question B:

Choice 1	Choice 2	Choice 3	No choice
27 (37%)	26 (34%)	14 (19%)	8 (10%)

For the question A, we notice that the usual presence of the graph of $y = 2x$ (A1, a straight line, correctly drawn by 92%) is related to the character of function attributed to the correspondence $x \rightarrow 2x$ (95%). Some students were in difficulties with the constant function (A2, the graph is correctly drawn by 81%; the character of function is attributed only by 72%). Remarkable difficulties were evident for Dirichlet's function (A3) and for Gelbaum's function (A4), whose it is impossible to draw the Cartesian graph.

For the question B, the (nearly) tie between the representation of the correspondence by symbolical way (B1) and by visual representation (B2) is interesting. It is important to remember that students learnt just two months before the test the representation of a relation by its Cartesian graph; the representation by arrows, often used to introduce didactically the concept of function, was chosen by 19% of the students.

2.3. JUSTIFICATIONS GIVEN BY STUDENTS (TEST 2)

Some students justified (in writing) their answers according to the concept of function; in particular, (question A2) we notice the frequent mistake that denies the character of function to the correspondence that to every $x \in \mathbf{R}$ associates 1 because it is not injective: this was the difficulty of 12 students out of 19 that answered incorrectly to the question A2.

Frequently the mistakes in the answers to the questions A3 and A4 depended upon the difficulty (or upon the impossibility) to draw the Cartesian graph of the correspondences: some students expressed heavy doubts caused by unusual relations ('I did not understand the exercise", Chiara; 'I never met a function like that, I did not know if it is possible to do it", Carlo), most of all with reference to Gelbaum's function (A4); the main part of the students that denied the character of function to Dirichlet's function and to Gelbaum's function underlined the impossibility to draw their Cartesian graphs: this justification is in the answers of 19 students of the 31 that did not consider a function Dirichlet's function (A3) and of 22 students (frequently the same students) of the 40 that did not consider a function Gelbaum's function (A4).

2.4. ANALYSIS OF JUSTIFICATIONS GIVEN BY SUDENTS AND CONCLUSIONS ABOUT TEST 2

We can conclude that the concept of function is often connected with the Cartesian graph of the relation examinated; for several students, this connection is sometimes predominant to decide the character of a function, it is the main feature that identifies a relation as a function. The possibility to draw its graph becomes the basic requisite: a relation can be considered a function when its Cartesian graph is a curve with some particular features (for example, a line identified by the equation $x = a$, $a \in \mathbf{R}$, must meet the curve only in one point).

This situation, intuitive so didactically important, must be controlled by the teacher ([6]): an exaggerated importance given to the visualization of a function could bring the students to a misunderstanding of the character of some relations (as Dirichlet's function and Gelbaum's function) that are refused as functions, not being visualized by a continuous curve.

2.5. METHOD AND RESULTS OF TEST 3

To verify the evolution of the situation pointed out in 3rd classes, previously shown, the same test, remembered in paragraph 2.2, was proposed to the students of three 5th classes *Liceo scientifico* (High School) in Treviso, Italy, total 66 students (their mathematical curricula were traditional). Results were:

Question A:

	Graph correctly drawn	It is a function	It is *not* a function	No answer
A1	66 (100%)	65 (98%)	0 (0%)	1 (2%)
A2	65 (98%)	58 (88%)	5 (7%)	3 (5%)
A3	0 (0%)	39 (59%)	18 (27%)	9 (14%)
A4	0 (0%)	19 (29%)	22 (33%)	25 (38%)

Question B:

Choice 1	Choice 2	Choice 3	No choice
22 (33%)	38 (58%)	4 (6%)	2 (3%)

These results (question A) seem to point out that the two school-years between students of 5th class and students of 3rd class **did not** improve the understanding of the concept of function, regarding to Dirichlet's function (A3) and Gelbaum's function (A4): they were correctly seen by 59% (3rd class: 46%) and by 29% (3rd class: 28%) of the students of 5th class.
For the question B, the tie between the choices B1 and B2, noticed for the students of 3rd class, is replaced by a clear preference of the students of 5th class for the Cartesian representation (58%, only 33% for the analytical representation). The representation by arrows, that in 3rd class was chosen by 19%, is not very important (and used) in the didactical practice in the *Liceo scientifico* (3rd, 4th and 5th classes) and it was remembered only by 6% of the students of 5th class.

([6]) It is spontaneous to remember Fischbein's note, previously quoted, about teachers' responsability (Fischbein, 1993, p. 156).

2.6. JUSTIFICATIONS GIVEN BY STUDENTS (TEST 3)

The mistake that denies the character of function to the correspondence that to every $x \in \mathbf{R}$ associates the real value 1 (A2), rather frequent for the students 3rd class, is rare for students of 5th class (only 3 students gave a wrong justification based upon the injectivity).

A remarkable part of students of 5th class tried to study Dirichlet's function (A3) and Gelbaum's function (A4) depending on their Cartesian graphs. In particular, 15 out of 18 students that did not consider a function Dirichlet's function and 16 students (frequently the same students) out of 22 students that did not consider a function Gelbaum's function underlined the impossibility to draw their Cartesian graphs.

Claudio's justification is interesting: 'The graph of the fourth relation does not exist, so I did not apply the rule by that a function must have a graph that can be met only once by a vertical straight line. Our teacher always requires this check". So the *didactical contract* has explicit clauses regarding visualization.

2.7. ANALYSIS OF JUSTIFICATIONS GIVEN BY STUDENTS AND CONCLUSIONS ABOUT TEST 3

As previously noticed about the results of the test proposed to the students of 3rd class, the concept of function is frequently connected with the Cartesian graph of the examinated relation. So the considerations expressed in paragraph 2.4 keep their importance for the test proposed to the students of 5th class.

We can conclude that the traditional didactical practice, strictly based upon the visualization of a function by its Cartesian graph, does not make easy the... liberation of the students from this technique of representation, didactically important, but *not exclusive*. Many students seem to identify a correspondence with its visualization (i. e. with its Cartesian graph) and this does not help them to remove the difficulties connected to the consideration of some (important) functions whose graphs are impossible to be drawn, as Dirichlet's function and Gelbaum's function.

3. GENERAL CONCLUSIONS

3.1. VISUALIZATION AS THE FINAL AIM...

Some features of the role of the visualization in mathematical education in the High School appear clearly by the considerations previously exposed.

It seems that the realization of a visual image of a concept (or of a procedure) is frequently considered the *final aim* of the whole learning course proposed to students from 3rd class to 5th class: the studies of Analytical Geometry and of Calculus bring students, in 5th class, to draw the Cartesian graph of *any* function (although, sometimes, the use of graphic calculators is now going to change this situation) (7). The emphasis that underlines this traditional exercise seems to be the natural conclusion of three years devoted to the development of the ability to visualize a real function.

We noticed that an excessive importance ascribed to the visualization of a function can be the cause of doubts, such as doubts caused by Dirichlet's and Gelbaum's function. We cannot forget that the identification of a function with its representation by Cartesian graph brings to an incorrect restriction of the relations to be considered: for example, students could be in difficulties to consider relations different from real functions of a real variable (8).

It is easy to notice that if didactical practice looks at the visualization as the main aim of the learning process, it can bring to inconveniences: the visualization could be excluded from deduction activities and pupils could make little use of a very important learning possibility (Duval, 1994a).

So we agree with A.H. Schoenfeld, who underlines the danger previously pointed out; he writes: «Pupils are competent when they deduce and they are competent when they construct, but they often sectorialize their knowledge... So a large sector of their knowledge remains unused and their performances in problem solving are much below to the level they could (and should) reach. An inappropriate sectorialization of activities of deduction and of activities of construction is a direct consequence of teaching» (Schoenfeld, 1986, p. 226).

3.2. ...OR GRAPHICAL REGISTER AS A POSSIBLE REGISTER OF REPRESENTATION, TO BE UTILIZED TOGETHER WITH OTHERS?

Considerations pointed out in the previous paragraph do not mean a limitation of the role of the visualization in mathematical education; we wish a full exploitation of the visualization: visual techniques are very important in the didactics of mathematics (Vinner, 1992; Duval, 1993; Fischbein, 1993).

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(7) We underline that our research is based upon two tests administred to about 140 students; to avoid over-interpretation, it would be necessary to investigate students' co nceptions by further tests, administred to many students.

(8) Of course, the important question of the difference between non-functional relations and functions is just *one* of the points in the learning of the concept of function. Moreover, other questions are very important such as: what kind of information do students take from the Cartesian graph of function? Are these informations correct or uncorrect? Do students use a graph (and how) when they are solving a problem about function?

The role of the visualization in mathematical education in High School (particularly referring to Italian *Liceo scientifico*) seems to be twisted out: it is sometimes considered suspiciously by students, while it could be used to make intuitive several mathematical procedures (for example, some basic algebraic techniques); on the contrary, the visualization becomes the main, final aim of the learning, for example in the case of the Cartesian graph of a function.

The theory of figural concepts by E. Fischbein, already remembered, plainly shows the "double nature", ideal, abstract and on the other hand real, of several mathematical objects (Fischbein, 1993); and this double nature should have important didactical implications, also for the utility of different registers of semiotic representation for the learning (Duval, 1993; Duval, 1995).

Our position is the following: visualization is a versatile, precious *starting-point* and an indispensable *travelling companion*, also for the positive, advantageous variety of the registers of representation, for the gradual and effective learning of techniques, of concepts, of "mathematical objects". The role of visualization should be transformed from the main and final aim (today sometimes suggested to students, perhaps tacitally imposed, included in some clauses of the *didactical contract*), to a powerful didactical mean (Schoenfeld, 1986; Duval, 1994a), really important for its great intuitive efficacy, an essential element of the stage of the learning that introduces pupils to abstraction.

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