Learning and Assessment in Mathematics and Science (2000),
Gagatsis, A.; Constantinou, C.P. & Kyriakides, L. (Eds.),
Department of Education, University of Cyprus, Nicosia, 171-196

CLASSICAL VERSUS VECTOR AND CARTESIAN GEOMETRY IN PROBLEM SOLVING IN GREECE AND IN ITALY

ATHANASIOS GAGATSIS
DEPARTMENT OF EDUCATION
UNIVERSITY OF CHYPRUS

GIORGIO T. BAGNI
NUCLEO DI RICERCA IN DIDATTICA
DELLA MATEMATICA, BOLOGNA

Summary. In this work we compared solving strategies for geometry problems in Greek and Italian High School. In the last year of Greek Lyceum (students aged 18 years), an analysis of written solutions to geometry exercises showed relatively low performances on vector methods. This can be caused either by false preconceptions with regards to the concept of vector, either by the influence of classical geometry teaching. As regards Italian High School (students aged 17-18 years), we noticed experimentally that many pupils approach geometry problems (that are not explicitly given in a Cartesian plane) by classical methods.

INTRODUCTION: TEACHING OF GEOMETRY IN GREECE AND IN ITALY

The teaching of geometry is a special didactic topic where social and historical issues can strongly influence the curriculum and pedagogy. In particular, an important point of teaching geometry is highly interesting: the existence, for several problems, of many different ways of approach: the classical method, the vectors method, the Cartesian one etc. In our opinion, it is really interesting to ask if the knowledge relating to one of these methods can improve or can become an obstacle relating to other methods. For instance, does the knowledge of the classical approach become an obstacle to understanding of the concept of vector? Is the use of static geometrical objects in classic geometry an obsacle to undestand and see the Cartesian method? Do appear in the solutions of students elements of different approaches at te same problem and why?

As regards solving strategies for geometry problems, national curricula of course deeply influence pupils' behaviour. In this work we are going to examine some strategies in Greek and Italian High School (secondary school), with reference to the different curricula.

In particular, in Greece, the study of classical (Euclidean) geometry is proposed to pupils aged 15-17 years; as regards pupils aged 18 years, last class of Lyceum, the study of vector methods is proposed (an historical analysis of curricula can be found in: Demetriadou & Gagatsis, 1995).

The great importance of a careful historical comprehension of geometry thought is underlined by E. Barbin, who writes: "Every reading implies a re-interpretation and every writing implies a re-appropriation of ideas, of knowledge. Re-interpretations and re-appropriations of geometry knowledge by means of basic geometry works are referred [...] to epistemological conceptions. And these conceptions must be, themselves, considered in their historical contexts" (Barbin, 1994, p. 157; translation is ours) ([1]).

In Italy, vector methods are not deeply treated in the High School: analytic geometry is diffused. Many students solve geometry problems by Cartesian (or trigonometric) method, but this particularly happens when problems themelves are explicitly given in the Cartesian plane; on the contrary, when pupils want to prove a theorem, they often prefer Euclidean method.

OUR EXPERIMENTAL RESEARCH IN GREEK AND ITALIAN SECONDARY SCHOOL (HIGH SCHOOL)

In a recent work (Gagatsis & Demetriadou, 1998) some resolutions of geometry problems given by Greek pupils (last year of Lyceum, pupils aged 18 years) are deeply examined (see moreover some previous experiences; for example: Gagatsis & Thomaidis, 1995; Demetriadou & Gagatsis, 1995).

The traditional teaching in Greek school, based upon classical (Euclidean) geometry, can cause a low performance as regards strategies based upon vector methods (these methods are considered deeply only in the last year of the Lyceum). In particular, in the quoted work the Authors considered some exercises that can be solved either by classical or by vector methods.

([1]) Purposes of a definition or of a proof can be different; ancient Chinese mathematicians considered two kinds of proof: *bian* (to convince) and *xiao* (to make aware, as underlined in: Barbin 1988; in the quoted work, the Author examine four editions of Euclid's *Elements*: *Les six premier livres des Éléments géométriques d'Euclide* by Peletier du Mans, 1557, *Nouveaux éléments de géométrie* by Arnauld, 1667, *Éléments de géométrie* by Clairaut, 1765, and *Éléments de géométrie* by Lacroix, according to edition of 1803). Let us remember moreover: Barbin, 1988 and 1991; Speranza, 1994. As regards proof, see: Furinghetti, 1992. G. Hanna considers the proof as "the definitive form of mathematical justification" (Hanna, 1997, p. 250).

As we shall see, the strong influence of classical teaching can be pointed out in the results: from interviews we can underline that often vector methods are considered with some perplexity; the greater part of the pupils work by classical methods and performance of the students that work by vector methods can be considered rather low. We notice that pupils are taught classical (Euclidean) geometry for four years (until they are 17 years old), and vector geometry is taught only to pupils of the final year of Lyceum (age 18; this consists also part of the examinations content for the entrance in the Greek universities: Gagatsis & Demetriadou, 1998). In the final year of Lyceum pupils are forced to use the concept of vector, although the teaching of this concept in mathematics courses during the previous years is either deficient or non-existent; moreover, some pupils have deficient or false conceptions (after the use in physics), in the field of geometry.

Gagatsis & Demetriadou (1998) write: "We would try to answer to the questions: what are the consequences of this deficient teaching of vectors on pupils of the last year of Lyceum behavior in solving geometry problems? Does the long experience in classical geometry still influence these pupils in solving geometry problems, even if this teaching is not recent nor particularly been emphasized, since it is not examined for the university entrance? Consequently, does the classical approach of geometry oppose vector methods?"

The quoted Authors examined Greek pupils (and of course we shall consider data achieved in their work); now we want to compare Greek and Italian situations. We must underline that, as above noticed, vector methods are not deeply considered in traditional curricula of Italian High School: in fact, pupils are taught classical (Euclidean) geometry for two years (until they are 16 years old), then analytical (Cartesian) geometry is taught to pupils in the three final years of *Liceo scientifico*. So, as regards Italian High School, the main comparision will take place between classical (Euclidean) methods and "other" methods (Cartesian and vector methods, sometimes trigonometric methods).

Data were collected from a sample of 361 pupils of the last year of Greek *Lyceum* (pupils aged 18), during 1996 (we shall consider results presented in: Gagatsis & Demetriadou, 1998), and from a sample of 223 pupils of Italian *Liceo Scientifico* (aged 17-18 years), during 1998. We used the following thirteen exercises (from this set, subsets from three or four exercises were given to each school; we underlined that exercises can be solved either by Euclidean or by vector, analytical or trigonometric methods):

Exercise 1: If two medians m_b and m_c of a triangle are perpendiculars, then we have: $m_b^2+m_c^2 = m_a^2$.

Exercise 2: Given an obtuse triangle ABC (A = 120°), prove that: $BC^2 = AC^2+$ $+AB^2+AC \cdot AB$.

Exercise 3: Given that A and B are the intersections of the two circles (K, R) and (L, r), prove that KL and AB are perpendicular.

Exercise 4: Two circles (K, R) and (L, r) touch each other externally at the point M. Given that AB is the common exterior tangent of the two circles, prove that AMB = 90°.

Exercise 5: Given that AD is the height of the isosceles triangle ABC with A = 90°, prove that $BC^2 = 2AC \cdot CD$.

Exercise 6: Given that M is the middle point of the side BC of a triangle ABC, prove that: $AB^2+AC^2 = 2AM^2+2MB^2$.

Exercise 7: In an isosceles triangle ABC (AB = AC), let D be a point on BC. Prove that: $AB^2–AD^2 = BD \cdot DC$.

Exercise 8: Given a triangle ABC with A = 150°, prove that: $a^2 = b^2+c^2+bc \cdot \sqrt{3}$.

Exercise 9: Given that AD is the height which corresponds to the hypotenuse of a rectangular triangle ABC (A = 90°), prove that $AD^2 = BD \cdot DC$.

Exercise 10: Given a quadrilateral ABCD with perpendicular diagonals AC and BD, prove that the sums of the squares of the opposite sides are equal, namely: $AB^2+DC^2 = AD^2+BC^2$.

Exercise 11: Every inscribed angle corresponding to a semi-circle is right.

Exercise 12: A rectangle ABCD has AB = 2AD; P is a point of the side DC such as $DP = 3/4 \cdot DC$, prove that BP is perpendicular to the diagonal AC.

Exercise 13: Given a parallelogram ABCD and the points E, Z of its diagonal such as AE = ZC = AC/4, prove that the quadrilateral EBZD is a parallelogram.

Tests were administered by classroom teachers during the normal school day. According to Gagatsis and Demetriadou (1998), the analysis of the results contains two parts: the first one concerns the success in solving geometry problems, methods used, and types of errors; the second one examines pupils' preferences as regards the different methods (advantages, disadvantages etc.) and whether these preferences were in agreement with methods actually used.

SOLVER TYPES AND ERRORS

We divided the solvers into three main categories:

Solver Types	*Chosen Method*
E	Pupils who used only Euclidean methods
V	Pupils who used only other methods: vector methods (Greek pupils) or analytical and vector methods (Italian pupils)
EV	Pupils who used both kinds of methods in different problems (in some problems Euclidean and in others vector methods) or a combination of them in the same exercise.

Let us underline once again that, as regards Italian High School, by "vector methods" we mean both vector and analytical methods.

In Malone et Al. (1980) we find some criteria for measuring problem-solving ability (Senk, 1985, used this model for secondary school geometry students in the United States). The model uses the following scoring scale:

Score 0-	Student writes nothing, or writes meaningless deductions.
Score 1-	Student approaches the problem by at least one valid deduction.
Score 2-	Student proceeds toward a rational solution by providing a chain of sufficient reasoning, however stops because of major errors or mis-interpretations.
Score 3-	Student has nearly solved the problem, but makes errors in notation, vocabulary or names of theorems.
Score 4-	Student gives a valid solution.

As clearly underlined in Gagatsis & Demetriadou (1998), in our research the problem-solving ability was not the main subject. Our priority was the ability in manipulating methods. For this reason a much smaller scale was used for the first two every categories of solvers (i.e. E, V):

Score	*Solution Stage*
0 (n)	Pupil gives no answer at all, writes wrong deductions to every exercise he deals with, or is not able to reach the end of a solution, although he makes correct steps.
1(n)	Pupil is merely successful; he is not able to solve sufficiently all the exercises he deals with, but solves sufficiently at least one exercise.
2(n)	Pupil provides valid solution to every problem he deals with, or nearly solves the problem with minor errors (in notation, vocabulary or names of theorems) which do not influence the correct result.

(by the variable n we mean E, Euclidean, or V, vector and other methods).

So scale 0 (n) includes the first three criteria of Malone et Al. model, scale 2(n) includes the two last cases of Malone et Al., and scale 1(n) is a middle situation between 0 (n) and 2(n). According to Gagatsis & Demetriadou (1998), we slightly modified this measuring model for the last category (EV): two more score types 4(EV) and 5(EV) were added in this category.

Score	*Solution Stage*
0(EV)	Failure both in vector and Euclidean methods.
1(EV)	Merely success in both methods.
2(EV)	Success in both methods.
3(EV)	Merely success in Euclidean methods and failure in vector methods.
4(EV)	Merely success in vector methods and failure in Euclidean methods.

As regards the type of errors found, we considered general errors concerning the solution procedure and, specifically, errors related to misconceptions as regards vector concept.

Once again according to Gagatsis & Demetriadou (1998), the classification of general errors was based on the empirical classification model introduced by Movshovitz-Hadar et Al. (1987; this model is empirical by the sense that "the investigation relied solely on data in students' answer books for a comprehensive examination"; the only theoretical assumption was that most of students' errors in high school mathematics "are not accidental and are derived by a quasi-logical process that somehow makes sense to the student": Movshovitz-Hadar & Al., pp. 3-4; the model consists of six descriptive categories of errors, which were identified in our research, except one category, "Unverified Solu tion"):

Categories of General Errors	*Analysis of errors*
Misused Data	Errors that deal with a discrepancy between the given data and the way that the examinee treated them. Characteristic elements of this category: neglecting given data and adding extraneous data, stating irrelevant requirements, assigning to some data a meaning that disagrees or is inconsistent with the text, incorrectly copying to the workbook.
Misinterpreted Language	Errors related to an incorrect translation of mathematical facts from one language to another. A characteristic element is the designation of a concept by a symbol traditionally designating another concept and operation with the symbol in its conventional use.

Logically Invalid Inference	Includes erroneous reasoning; e.g. an unjustified jump in a logical inference without providing the necessary sequence of arguments. We have also included the following cases: proving that p = p when it is asked to prove that p = q, and concluding that p implies q by providing as argument the validity of q .
Distorted Theorem or Definition	Errors in applying a theorem outside its conditions, or an imprecise citation of a recognizable theorem or formula.
Technical Errors	Computational errors and errors in mathematical symbols and algorithms.

Some examples for each category of general errors are provided in: Gagatsis and Demetriadou, 1998.

Except general errors, we considered errors resulted from misconceptions of procedures and concepts related directly to the concept of vector (see: Demetriadou, 1994; Demetriadou and Gagatsis, 1995):

Categories of Vector Errors	*Analysis of errors*
Vector equivalent to a line segment.	Errors dealing with the misconception of a vector as a concept equivalent or very close to the concept of a line. In this case among the features of a vector (magnitude, sense, orientation) magnitude seems to prevail in pupil's mind. Characteristic elements: consideration of a line segment as a vector, when a relation with vectors is directly converted to a relation of their magnitudes, when vectors of equal magnitudes are considered as equal, when vectors are considered as equal to their magnitudes, when the convention of parallelism or perpendicularity between vectors is used for line segments.
Sense Errors	Errors related to misconceptions about the concept of sense, or where sense is not considered as feature of a vector. Characteristic elements: errors in vector addition, errors in dot product, opposite vectors are equal.
Errors in Vector Addition and Subtraction	Errors related to wrong procedure of these operations. A characteristic element is the wrong replacement of a vector by the sum of two vectors which are not its components.
Errors in Dot Product	Erroneous application of the procedure of dot product. Characteristic errors as regards angle between two vectors.
Errors in Using Coordinates	Errors in expressing a vector by the use of coordinates, and errors in dot product in the case that coordinates are used.

Some examples for each category of vector errors are provided in: Gagatsis and Demetriadou, 1998.

We asked to the the pupils to answer to the following questionnaire, after they had finished with the problem solving procedure:

- *Is there another method for the solution of these problems?*
- *Why have you preferred this concrete method and not another one?*
- *According to your opinion which is the best method and why?*
- *What difficulties are you faced with, for each method? Namely, what forced you to abandon a method, if something like that has happened, or what has prevented you in reaching a final solution?*

Pupils (361 Greek pupils and 223 Italian pupils) made positive or negative commands for the one or for both methods (let us remember that, as regards Italian High School, by "vector methods" we mean methods based upon vector geometry and Cartesian analytical geometry).

As regards Euclidean methods, we indicate:

Positive Commands for Euclidean Geometry	*Characteristic Responses*
(Simplicity) Quite simple	"easy", "brief", "comprehensible", "accessible".
(Long Experience) Pupils have a long time experience as concern this method	"During the last five years we used this method in solving exercises". "It's more familiar".
(Aid of Theorems) The existence of many theorems make it more easy and safe	"It's more organized and methodical, s ince it is based upon theorems; you know what you want and where you go". 'This method is simpler because it is based on known theorems".
Interesting (imagination, logic, clever thoughts)	"It gives me more mathematical satisfaction, because it excites my imagination". "It's clever" and "enjoya ble". "It's based on logical arguments".
(Unique in Pupils' Mind) It was the only method been remembered	"There is no other method". "I could not find another method". "This method came first in my mind".

Negative Commands about Euclidean Geometry	*Characteristic Responses*
(A lot of Theory) Large piece of knowledge and figure difficulties	"Auxiliary lines are needed, as well as a lot of theorems and types".
Complicated thought	'Difficult thought", 'imagination is needed". 'In some exercises you must observe very carefully. Many times the figure misleads you".
(Not recent) Pupils have forgotten the relative theory	'I cannot remember exactly what we've been taught in the previous years of Lyceum". 'I do not remember some theorems". 'I have forgotten several theorems and I am not sure about some basic definitions, too".

As regards vector methods, we indicate:

Positive Commands about Vector Geometry	*Characteristic Responses*
(Recent) Recent and useful for the entrance in the universities	'Vector Geometry is a new chapter and I find it interesting". 'It's more familiar to me". 'It's part of the content for the last year of Lyceum, and for the examinations for university enrollment".
(Methodology) Effective and standardized	'Comprehensible for complicated exercises". 'General. effective". 'Simple relations and simple operations are used". 'Less knowledge (types and theorems) is needed". 'It does not demand imagination". 'Some combinations with vectors are needed". 'No auxiliary lines are needed". "There is no need to localize something in the figure". 'It is a modern and powerful method".
(Unique in Pupils' Mind) It was the only method in the pupils' mind	'There is no other method". 'I could not find another method".
(More Favorable) Pupils fill pleasure in exploring the new method	'It's better than Euclidean method. Unfortunately I have discovered it too late!" "This method analyzes the problems better than the Euclidean method".

Negative Commands about Vector Geometry	*Characteristic Responses*
(Lack of Experience) Pupils are not experienced on vector methods	"Vectors scare me, since I have not a clear image of them in my mind". "We do not know vectors well, since these have not been taught during the previous years". "Vectors are easier (than classical, Euclidean geometry), but we've been taught about them for shorter time".
Complicated (Types, operations, vector sense)	"The types are more standardized, but I do not know how to continue every time (or they are easily forgotten)". "Inconvenient types". "There are many and difficult operations". "In most cases it is not so brief ". "Vectors need more details". "Vector senses confuse me, therefore I preferred a more safe method".

(There were also pupils who supported both methods: Gagatsis & Demetriadou, 1998).

So we discriminated five types of pupils as concern their attitude towards the two methods:

(+E)	Pupil signifies his preference to Euclidean method or/and mentions the advantages of this method.
(- E)	Pupil mentions the disadvantages of Euclidean method.
(+V)	Pupil signifies his preference to vector methods or/and mentions the advantages of this method.
(-V)	Pupil mentions the disadvantages of vector methods.
(+EV)	Pupil signifies his preference to both methods.

According to Demetriadou and Gagatsis (1998), each response corresponds either to one of the above trends or is a combination of trends (for example, a pupil was characterized as being of the compound type (+E)(-E) if he mentioned some disadvantages of Euclidean method although he had signified his preference to this method).

Method supporters	*Attitude towards geometry methods*
Euclidean supporter	Pupil has a positive attitude towards Euclidean method.
Vector supporter	Pupil has a positive attitude towards vector methods.
Both methods supporter	Pupil has a positive attitude towards both methods.

RESULTS OF THE TEST

Let us give the classification of pupils as concern the method used by them.

As regards Greek High School, we observe that most pupils decided to use both methods (47%), then follow pupils which dealt exclusively with Euclidean methods (39%), while only a few pupils (14%) chose to deal exclusively with vector methods. As regards Italian High School, we observe that most pupils decided to use Euclidean methods (52%), then follow pupils which dealt with both methods (39%) and once again only a few pupils (9%) dealt exclusively with vector methods.

Number (n) and Percentage of Pupils Using a Geometry Method				
	Greek High School		Italian High School	
Solver Types	n	%	n	%
Solver Type E	139	39	116	52
Solver Type V	51	14	20	9
Solver Type EV	171	47	87	39
Total	361	100	223	100

The next question was the grade of success for each solver category.

As regards Greek High School, we can verify that Euclidean solvers are most successful than vector solvers; as concern pupils using both methods, they seem to be more successful in Euclidean methods. A similar trend can be pointed out as regards Italian High School.

Percentage of Scores for Each Solver Type						
	Percentage Getting Each Score					
	Greek High School			Italian High School		
Score	Solver E	Solver V	Solver EV	Solver E	Solver V	Solver EV
0(n)	16	25	14	9	30	10
1(n)	37	41	7	49	50	14
2(n)	47	33	40	42	20	8
3(n)	-	-	33	-	-	58
4(n)	-	-	5	-	-	9

By the variable n we mean E (Euclidean), V (vector), or EV (Euclidean and Vector).

We can notice that there are more pupils using Euclidean methods than vector ones, and that these pupils are also more successful in problem solving than pupils using vector methods. Even in the case when pupils choose to use both methods, they are more successful in exercises solved by classical methods.

Let us now consider errors: as regards Greek High School, we dealt with 407 errors, 282 general and 125 vector errors. We shall observe that general errors constitute the majority for types E and EV (99% and 65% respectively), while vector errors constitute the majority for type V (61%).

Number and Percentage of Errors for Each Solver Type – Greek High School						
Error	Solver Type E		Solver Type V		Solver Type EV	
Categories	n	%	n	%	n	%
General Errors	103	99	26	39	153	65
Vector Errors	1	1	41	61	81	35
Total	104	100	67	100	234	100

As regards Italian High School, we found 191 errors that are classified in the following table (let us remember that by "vector errors" we mean errors related to vectors and to use of coordinates in Cartesian methods).

Number and Percentage of Errors for Each Solver Type – Italian High School						
Error	Solver Type E		Solver Type V		Solver Type EV	
Categories	n	%	n	%	n	%
General Errors	83	100	11	58	68	76
Vector Errors	0	0	8	42	21	24
Total	83	100	19	100	89	100

We divided general errors of type EV in two sub-categories, errors found in problems solved either by classical or vector methods, in order to make a convenient comparison with errors of types E and V. We noticed in Greek and in Italian High School that it seems that solvers of any type show in many cases similar behavior in committing errors, when they deal with the same method.

Percentage of General Errors for Each Solver Type – Greek High School				
	Percentage of Errors			
Category	Type E	Type V	Type EV	
			E Method	V Method
Misused Data	30	8	21	14
Misinterpreted Language	5	42	7	44
Logically Invalid Inference	28	4	27	9
Distorted Theorem or Definition	30	15	43	17
Technical Errors	7	31	1	15

Percentage of General Errors for Each Solver Type – Italian High School				
	Percentage of Errors			
Category	Type E	Type V	Type EV	
			E Method	V Method
Misused Data	18	9	5	20
Misinterpreted Language	2	9	9	4
Logically Invalid Inference	43	27	16	8
Distorted Theorem or Definition	10	0	21	4
Technical Errors	27	55	49	64

As concern Euclidean geometry, it seems that errors resulted from insufficient knowledge of theory (Distorted Theorem or Definition) prevail among general errors (30% for E-type and 43% for EV-type), in Greek High School. Gagatsis and Demetriadou underline that "the difference of 13% shows that EV-pupils have more difficulties with theory, and could be a reason for which pupils try also vector methods, since they do not fill as safe as E-pupils when using classical methods" (Gagatsis & Demetriadou, 1998). As regards vector geometry, in Greek High School, misinterpreted language-errors prevail among general errors (42% for V-type and 44% for EV-type).

As regards Italian High School, errors resulted from logically invalid inference prevail among general errors (E-type: 43%); as regards EV-type, the greater part of general errors is connected to technical errors (49%).

We agree with Gagatsis & Demetriadou (1998), who underline that traditional geometry is not only memorizing propositions, precise definitions, and proofs of theorems (as regard verbal, logical, visual, and drawing skills connected with classical geometry, see: Hoffer, 1981; as regards memorizing obligation demanded by Euclidean geometry, see: Kimball, 1954: it has a negative influence on pupils' achievement in traditional geometry; as regard difficulties among secondary school geometry students, see for instance: Senk, 1985, and Carpenter et Al., 1981). The main feature which characterize Euclidean geometry is a logical procedure, and in our research we found errors resulted from a logically invalid procedure (see once again the last table, referred to Italian High School). Figure is also another cause of difficulties for pupils.

As regards vector errors in Greek High School, we dealt altogether with 125 errors: 1 for solver type E, 41 for solvers type V and 81 for type EV (3 errors for Euclidean methods and 80 for vector methods); in Italian High School we dealt with 29 vector errors: no errors for solvers type E, 8 for solvers type V and 21 for type EV (all of them for vector methods).

The following tables present an analysis of general errors for the three solver types (Greek and Italian High School).

Percentage of Vector Errors for Each Solver Type – Greek High School				
	Percentage of Errors			
Category	Type E	Type V	Type EV	
			E Method	V Method
Vector as a line segment	100	49	100	57
Sense Errors	0	34	0	25
Addition and Subtraction Errors	0	12	0	11
Errors in Dot Product	0	0	0	4
Errors in Using Coordinates	0	5	0	4

Percentage of Vector Errors for Each Solver Type – Italian High School				
	Percentage of Errors			
Category	Type E	Type V	Type EV	
			E Method	V Method
Vector as a line segment	0	12	0	14
Sense Errors	0	0	0	5
Addition and Subtraction Errors	0	12	0	14
Errors in Dot Product	0	0	0	0
Errors in Using Coordinates	0	75	0	67

Clearly vector errors are substantially connected with vector solvers, both in Greek and in Italian High School. Data above given are particularly interesting as regards Greek High School: the misconception that vector is equivalent to a line segment prevails among vector errors (49% for V-type and 57% for EV-type), while sense errors come after (34% for V-type and 25% for EV-type). Let us remember that the teaching and comprehension of the concept of vector as regards Greek secondary pupils (aged 15, 16, and 17 years), has been studied in some previous researches (Demetriadou, 1994; Demetriadou and Gagatsis, 1995): the results showed relatively low comprehension, which was partly attributed to the peculiarity in teaching vectors at Greek High School. These results help to explain some errors of Greek pupils (aged 18 years) in vector geometry.

Cocerning Italian High School, collected data are not very interesting: as above underlined, vector methods are not among main geometry methods in Italian traditional High School curricula, so the greater part of errors would be connected to Cartesian methods.

Summarizing, we could say (in agreement with Gagatsis & Demetriadou, 1998) that *errors resulted from insufficient knowledge and manipulation of* *theory, as well as errors of logical procedure characterize Euclidean solvers* (as regards Greek and Italian High School)*, while vector solvers make more errors resulted from misconceptions about the concept of vector* (particularly as regards Greek High School).

PUPILS' OPINIONS ABOUT GEOMETRY METHODS

We investigated pupils' opinions and dispositions as regards different approaches, and the possible influence of this disposition in the choice of a method for problem solving.

Let us summarise results achieved in the following table.

Number (n) and Percentage of Pupils Supporting a Geometry Method				
	Greek High School		Italian High School	
Method Supporters	n	%	n	%
Euclidean (+E)	157	43	83	37
Vector (+V)	85	24	13	6
Both Methods (+EV)	119	33	127	57
Total	361	100	223	100

In order to verify whether pupils' preferences are in agreement with their choices as regards the geometry approaches, we shall present the percentages of different method-supporters as they are divided among the set of the three solver types: we underline that most of the solvers in all three types seem to prefer the method which they actually used; so their choices were not accidental and several pupils seem to be aware of the advantages of the chosen method.

Percentage of Method Supporters for Each Solver Type						
	Percentage of Solver Types					
	Greek High School			Italian High School		
M. Supporters	Type E	Type V	T. EV	Type E	Type V	T. EV
Euclidean (+E)	83	12	21	60	10	13
Vector (+V)	3	78	24	4	35	1
Both m. (+EV)	14	10	55	36	55	86

Let us now give the inverse correspondence between chosen and most preferable method.

Percentage of Solver Types Supporting Each Method – Greek High School			
	Percentage of Method Supporters		
Solver Types	Euclidean (+E)	Vector (+V)	Both methods (+EV)
Type E	73	5	17
Type V	4	47	4
Type EV	23	48	79

Percentage of Solver Types Supporting Each Method – Italian High School			
	Percentage of Method Supporters		
Solver Types	Euclidean (+E)	Vector (+V)	Both methods (+EV)
Type E	84	38	32
Type V	2	54	9
Type EV	14	8	59

Concluding, we can notice that Euclidean and vector solvers chose the method which they prefer. Particularly as regards Greek High School, concerning the supporters of vector methods, it seems that their preference is not always strong enough to make them use this method: is the influence of classical geometry so strong that it is used as a refuge, in spite of its disadvantages as they are recognized by pupils themselves?

Percentage of Positive Features for Euclidean Method						
	Percent of Positive Features					
	Greek High School			Italian High School		
Advantages of Euclidean Methods (+E)	Type E	Type V	Type EV	Type E	Type V	Type EV
Simplicity	47	20	51	16	50	45
Long experience	21	40	19	24	0	5
Aid of theorems	9	40	12	54	50	45
Unique in pupils' mind	21	0	4	3	0	0
Interesting	6	0	14	3	0	0

Percentage of Negative Features for Euclidean Method						
	Percentage of Negative Features					
	Greek High School			Italian High School		
Disadvantages of Euclidean Methods (-E)	Type E	Type V	Type EV	Type E	Type V	Type EV
A lot of theory	75	50	45	50	100	67
Not recent	25	50	27	50	0	33
Complicated	0	0	27	0	0	0

So responses of pupils gave interesting information as regards their ideas about characteristics of different approach. Of course, it is easy to underline that some of the positive features of the one method operate as negative for the other, both in Greek and in Italian High School.

Percentage of Positive Features for Vector Methods						
	Percent of Positive Features					
	Greek High School			Italian High School		
Advantages of Vector Methods (+V)	Type E	Type V	Type EV	Type E	Type V	Type EV
Methodology	50	44	62	20	0	0
More favorable	44	29	17	20	29	0
Recent	6	18	20	60	71	100
Unique in pupils' mind	0	9	1	0	0	0

Percentage of Negative Features for Vector Methods						
	Percent of Positive Features					
	Greek High School			Italian High School		
Disadvantages of Vector Methods (-V)	Type E	Type V	Type EV	Type E	Type V	Type EV
Complicated	50	0	77	0	0	0
Lack of Experience	50	0	23	100	0	100

CONCLUSIONS

Gagatsis and Demetriadou (1998) correctly divide considered Greek pupils into three main groups: in the first one there are the students strongly influenced from previous school teaching; so mistakes can be related to weak knowledge of the theory or to invalid logic inferences (in some cases pupils express the pleasure they feel when recalling classical methods; for example: 'It is more interesting; it's a pity it is not taught in the last year of Lyceum!"). In the second category there are pupils that use vector methods in solving geometry problems; mistakes are often related to vector symbology and to operations. In the third category we can consider pupils influenced by both approaches: however these pupils seem to prefer methods referred to classical geometry.

As regards Italian High School, we underlined that, according to the main traditional curriculum, pupils are taught classical (Euclidean) geometry for two years (until they are 16 years old) and then Cartesian geometry is taught to pupils of three final years of *Liceo Scientifico* (pupils aged 16-19 years);

however when Italian students deal with geometry problems (i.e. problems not *explicitally* given in a Cartesian plane), the greater part of them prefers to approach problems by the well-known classical (Euclidean) method: so we could say that the context in which exercises themselves are given is very important as regards the choice of the method and the solving strategy.

We can conclude that Greek and Italian pupils' behavior is not very different (of course, we can state this with reference to considered samples): Euclidean geometry's simplicity, the remarkable aid of many theorems and definitions are positive commands explicitally recognized by several pupils; but relatively low performances in vector methods show that probably national curricula can be strongly improved.

References

Barbin, E. (1988), La démonstration mathématique: significations épistémologiques et questions didactiques: *Bulletin de l'APMEP*, 366. Moreover: La dimostrazione matematica: significati epistemologici e questioni didattiche: *Quaderni di lavoro n. 10*, Istituto Filippin, Paderno del Grappa.

Barbin, E. (1991), La démonstration mathématique: histoire, épistémologie et enseignement: *Actes des 2èmes journées Paul Langevin*, Université de Brest.

Barbin, E. (1994), Sur la conception des savoirs géométriques dans les Éléments de géométrie: Gagatsis, A. (ed.), Histoire et enseignement des Mathématiques: *Cahiers de didactique des Mathématiques*, Thessaloniki, 14-15, 135-158.

Carpenter, T.P.; Corbitt, M.K.; Kepner, H. S.; Lindquist M.M. & Reys, R.E. (1980), Geometry, Corbitt, M.K. (Ed.), *Results from the Second Mathematics Assessment of the National Assessment of Educational Progress*, The National Council of Teachers of Mathematics, Inc., Reston.

Demetriadou, H. (1994) *The Teaching of the concept of vector in Greece- Some Remarks on the history of this concept and on the errors to the Greek pupils*. M.A. Dissertation, University of Surrey, Roehampton Institute, London.

Demetriadou, H. & Gagatsis, A. (1995), Teaching and learning problems on the concept of vector in Greek secondary education: Gagatsis, A. (ed.), *Didactics and history of mathematics*, Erasmus ICP 94-G-2011/11, Thessaloniki.

Furinghetti, F. (ed.) (1992), *Definire, argomentare e dimostrare nel biennio e nel triennio: opinioni, esperienze e risultati di ricerche a confronto*, Atti del secondo Internucleo della Scuola secondaria superiore, Progetto strategico del CNR: Tecnologie e innovazioni didattiche, Quaderno n. 13.

Gagatsis, A. & Thomaidis, J. (1995), Eine Studie zur historischen Entwicklung und didactischen Transposition des Begriffs "absoluter Betrag": *Journal für Mathematik-Didactik*, 16, 1/2, 3-46.

Hanna, G. (1997), Il valore permanente della dimostrazione: *La matematica e la sua didattica*, 3, 236-252.

Hoffer, A. (1981), Geometry is more than proof, *Mathematics Teacher*, 74 (1), 11-18.

Kimball, R.B. (1954), Direct vs. indirect memory in geometry, *Mathematics Teacher* , XLVII (1), 13-15.

Malone, J.A., Douglas, G.A., Kissane, B.V., & Mortlock, R.S. (1980), Measuring problem-solving ability, Krulik, S. & Reys, R.E. (Eds.), *Problem Solving in School Mathematics*, 1980 Yearbook of the National Council of Teachers of Mathematics, Reston, The Council.

Movshovitz-Hadar, N.; Zaslavsky, O., & Inbar, S. (1987). An empirical classification model for errors in high school mathematics. *Journal for Research in Mathematics Education* , 18(1), 3-14.

Senk, S.L. (1985). An empirical classification model for errors in high school mathematics. *Mathematics Teacher* , 78(6), 448-456.

Speranza, F. (1994), A la recherche d'une philosophie de la mathématique idoine pour la didactique: Gagatsis, A. (ed.), Histoire et enseignement des Mathématiques: *Cahiers de didactique des Mathématiques*, Thessaloniki, 14-15, 187-194.