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Trigonometric functions: learning and *didactical contract*

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Summary. In this paper the influence of the *didactical contract* on the students in the learning of mathematics is investigated, referred to the introduction of trigonometric functions, in Italian High School (*Liceo scientifico*, pupils aged 16-19 years).

PREFACE

In High School, the influence of the *didactical contract* on the students is remarkable: we shall examinate a "chapter" of the mathematical education (referred to Italian *Liceo scientifico*, pupils aged 16-17 years) and we shall expose some considerations about its learning [Brousseau, 1987].

Frequently, trigonometric functions are introduced initially referred to the values sin$x$, cos$x$, tan$x$... assumed for values of x of common use: sin$x$, cos$x$, tan$x$... can be easily calculated when x is put

$$0, \pi/6, \pi/4, \pi/3, \pi/2..., \pi, ...$$

This introduction can cause heavy misunderstandings about trigonometric functions: for example, students can think that sin$x$, cos$x$, tan$x$, ... can be calculated *only* for some particular values assumed by x.

The resolution of a trigonometric equation, in the traditional mathematical curriculum of High School, is a very common exercise. However the resolution of any trigonometric equation would be prefaced by the introduction of the inverse functions expressed by $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, considered mathematically exacting. But this is not frequent: sometimes (and... not rarely), the resolution of easy trigonometric equations is one of the exercises suggested to the students just after the introduction of the functions expressed by sin$x$, cos$x$, tan$x$.

For example, students can be asked to find x (being $0\le x\le 2\pi$) such that

$$\sin x = 1/2$$

They can remember the following well-known table:

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π ...
$\sin x$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0 ...
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1 ...
$\tan x$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	(no)	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0 ...

So they conclude:

$$\sin x = 1/2 \qquad (\text{being } 0 \le x \le 2\pi) \qquad \Rightarrow \qquad x = \pi/6 \ \vee \ x = 5\pi/6$$

If the pupils know the periods of trigonometric functions, they can write:

$$x = \pi/6 + 2k\pi \ \vee \ x = 5\pi/6 + 2k\pi$$

being k any integral number.

The following equation is surely harder:

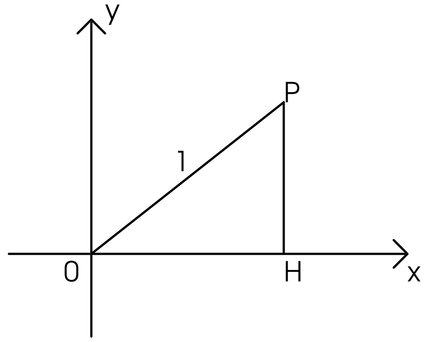
$$\sin x = 1/3$$

The value 1/3 *is not among the values in the table above given*, and the resolution of this equation cannot be based upon the table, but it requests, for example, the use of a scientific calculator.

From that, any student would think: what is the meaning of this exercise? Does it exist, *can it exist* a value of x such that $\sin x = 1/3$?

METHOD OF TEST

The following test was proposed to students belonging to two 4th classes of a *Liceo scientifico* (High School) in Treviso, Italy, total 67 pupils; their mathematical curricula were standard; they knew trigonometric functions $\sin x$, $\cos x$, $\tan x$ and their periods. For example, $x \to \sin x$ was introduced by relating to every HÔP the measure of PH, (the measure of OP is 1); $x \to \cos x$ was introduced by relating to every HÔP the measure of OH.



The teacher underlined that the domain of $x \to \sin x$ and of $x \to \cos x$ is the set **R** of the real numbers. Values of $\tan x$ were introduced by $\tan x = \frac{\sin x}{\cos x}$, being $\cos x \neq 0$. Cartesian diagrams of $y = \sin x$, of $y = \cos x$ and of $y = \tan x$ were introduced. Values of $\sin x$, $\cos x$, $\tan x$ were resumed (being x: 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, ..., π, ..., 2π) in the table above remembered. Students did not know $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, ...

Test:

Find $x \in \mathbf{R}$ such that:

a) $\sin x = -1/2$	b) $\cos x = 1/2$
c) $\sin x = 1/3$	d) $\tan x = 2$
e) $\sin x = \pi/3$	f) $\cos x = \pi/2$
g) $\sin x = \sqrt{3}$	h) $\cos x = -\sqrt{3}/3$

Time: 30 minutes. No books or calculators were allowed.

RESULTS OF THE TEST

(A): $\sin x = -1/2$.

Total.	Correct answers (with period):	41	(61%);
	Correct but partial answers:	14	(21%);
	Wrong answers:	8	(12%);
	No answer:	4	(6%);

(B): $\cos x = 1/2$.

Total.	Correct answers (with period):	45	(67%);
	Correct but partial answers:	10	(15%);
	Wrong answers:	7	(11%);
	No answer:	5	(7%);

(C): $\sin x = 1/3$.

Total.	Correct answers (x written by α, β, by graphic method) and with period:	12	(18%);
	Correct but partial answers:	16	(24%);
	Wrong answers:	18	(27%);
	No answers:	21	(31%);

(D): $\tan x = 2$.

Total.	Correct answers (x written by α, β, by graphic method) and with period:	11	(16%);
	Correct but partial answers:	10	(15%);
	Wrong answers:	20	(30%);
	No answer:	26	(39%);

(E): $\sin x = \pi/3$.

Total.	Correct answers:	25	(37%);
	Wrong answers:	18	(27%);
	No answer:	24	(36%);

(F): $\cos x = \pi/2$.

Total.	Correct answers:	29	(44%);
	Wrong answers:	19	(28%);
	No answer:	19	(28%);

(G): $\sin x = \sqrt{3}$.

Total.	Correct answers:	31	(46%);
	Wrong answers:	19	(28%);
	No answer:	17	(26%);

(H): $\cos x = -\sqrt{3}/3$.

Total.	Correct answers (x written by α, β, by graphic method) and with period:	13	(20%);
	Correct but partial answers:	10	(15%);
	Wrong answers:	19	(28%);
	No answer:	25	(37%).

CONSIDERATIONS ABOUT RESULTS

- Several students can solve *standard* exercises (a), (b), but they cannot solve exercises in which results are not the common value of x, so (c) and (d). Many students did not answer to question (c) (31%) or to question (d) (39%); several students, moreover, answered *"impossible"* to one of those questions (or both).

The following mistake is rather frequent (7 times) and interesting:

$\tan x = 2 \Rightarrow x = \pi/3$, or: $x = \pi/3+k\pi$

Note that $\sqrt{3}$, the value assumed by $\tan x$ being $x = \pi/3$, is very close to the value 2.

- Results about (e), (f) are interesting; many students showed difficulties (36% and 28% did not answer).

The following mistakes are interesting:

$\sin x = \pi/3 \Rightarrow x = \sqrt{3}/2$ (11 times)

$\cos x = \pi/2 \Rightarrow x = 0$ (14 times)

sometimes with the period ($\sin x = \pi/3 \Rightarrow x = \sqrt{3}/2+2k\pi$ and $\cos x = \pi/2 \Rightarrow x = 0+2k\pi$); anyway, several students wanted to find an answer, so they associated uncorrectly the values.

- Results about (g), (h) are interesting, too (26% and 37% did not answer) Many students (approximately 20%) answered *"impossible"* to one of those questions (or both).

The following mistakes are interesting:

$\sin x = \sqrt{3} \Rightarrow x = \pi/3$, or: $x = \pi/3+k\pi$ (11 times)

$\cos x = -\sqrt{3}/3 \Rightarrow x = -\pi/6$, or: $x = -\pi/6+k\pi$ (9 times)

So some students confused values of tan$x$ and values of sen$x$, cos$x$.

JUSTIFICATIONS GIVEN BY STUDENTS

Several students gave interesting justifications, particularly referred to the answers to the questions (d), (e), (g) (the "table", frequently remembered, is the table, above mentioned, in which students can read the values of the trigonometric functions for $x = 0, \pi/6, \pi/4, \pi/3, \pi/2, ..., \pi, ..., 2\pi$).

Mistake: $\tan x = 2 \Rightarrow x$ does not exist (question d)

Tullio stated that he did not find the value 2 in the table, so he thought that no x is such that $\tan x = 2$.

Marika stated that sin$x$ and cos$x$ cannot be greater than 1, so tan$x$, too, cannot be greater than 1.

Mistake: $\tan x = 2 \Rightarrow x = \pi/3$, or: $x = \pi/3+k\pi$ (question d)

Alberto noted: "I immediately remembered that tan($\pi/4$) = 1; in the table, after the value $\pi/4$, there is the value $\pi/3$, so I thought that tan($\pi/3$) was 2".

Mistake: $\sin x = \pi/3 \Rightarrow x = \sqrt{3}/2$ (question e)

Maria Chiara stated: "In my opinion, it is really strange that an exercise have no solutions".

Leopoldo remembered that in the table the value $\pi/3$ is associated to the value $\sqrt{3}/2$: so he thought that $\sin(\sqrt{3}/2) = \pi/3$.

Mila said she 'visualized' in her minds the table, so she "saw" $\pi/3$ close to $\sqrt{3}/2$.

Enrico noted: "I was sure that $\pi/3$ is associated to $\sqrt{3}/2$, and I made a bad mistake. I shall remember that if an exercise seems too much easy, it is always dangerous!"

Tullio noted: "I know that π is 3,14 [and 3,14/3 is greater than 1], but I thought that π is 180°".

Mistake: $\sin x = \sqrt{3} \Rightarrow x = \pi/3$, or: $x = \pi/3+k\pi$ (question g)

Enrico admited that he associated $\sqrt{3}$ to $\pi/3$ because $\tan(\pi/3) = \sqrt{3}$.

All the students accepted the corrections.

CONCLUSIONS

- About the answers to the questions (c) and (d), the existence of a value x such that $\tan x = 2$ was often related to the presence of the value 2 in the table. Alberto's mistake, too, was based upon a wrong use of the table (Alberto *inserts* the value 2 in the table, after the value 1). Marika's mistake was not based upon the wrong use of the table: she remembered a property of $\sin x$ and of $\cos x$, and she referred it to $\tan x$, too.
- About the answers to the questions (e) and (f), the *didactical contract* forced the student to find *always* a solution, for *every* exercise (Maria Chiara stated: "it is really strange that an exercise have no solutions"): so many students associated, for example, $\pi/3$ and $\sqrt{3}/2$ or $\pi/2$ and 0.
- Moreover, two statements are remarkable: the role of visual memory (Mila said that it caused her mistake) and Tullio's answer, in which π is considered a *real number* and the *measure of an angle* (in Tullio's words: "3,14" and "180°") [Duval, 1993] [Fischbein, 1993].
- About the answers to the questions (g) and (h), the reference to $\tan x$ is clear (see Enrico's justification and the period $+k\pi$): once more, the *didactical contract* obliged some students to look for a correspondence in which the given values are present.

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