

“SIMPLE” RULES AND GENERAL RULES IN SOME HIGH SCHOOL STUDENTS’ MISTAKES

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Summary. In this paper some common mistakes are investigated, referred to linear mappings and to the solution of algebraic equations, with reference to High School students (students aged 16-19 years). We conclude that pupils sometimes improperly extend «simple» rules, and this is caused by algebraic weakness and by affective elements, too. As regards strategies against misguided generalisations, we underline that the effect of counterexamples with pupils is frequently weak, since often they are not able to interpret counterexamples in an adequate way.

1. ARE ALL FUNCTIONS LINEAR MAPPINGS?

As regards «certain side routes which can be taken by a learner», A. Sfard writes: «The student may manipulate a concept through a certain prototype – for example, the data collected by Markovitz et Al., 1986, show that beginners tend to imagine linear mappings whenever the notion of function is mentioned» (Sfard, 1991, p. 21).

Indeed the property of a function to be a linear mapping seems to be a fundamental rule for several students. Every mathematics teacher knows mistakes like:

$$\begin{array}{lll} (a\pm b)^2 = a^2\pm b^2 & \text{or:} & (a\pm b)^3 = a^3\pm b^3 & \text{etc.} \\ \sqrt{a\pm b} = \sqrt{a}\pm\sqrt{b} & \text{or:} & \sqrt[3]{a\pm b} = \sqrt[3]{a}\pm\sqrt[3]{b} & \text{etc.} \\ \sin(a\pm b) = \sin a\pm\sin b & \text{or:} & \cos(a\pm b) = \cos a\pm\cos b & \\ \log_e(a\pm b) = \log_e a\pm\log_e b & & & \end{array}$$

Really they are not rare: their presence can be underlined in students’ protocols in several school-levels (as regards algebraic misguided generalisations, we indicate: Tietze, 1988 and Malle, 1993; for High School students, too, pupils aged 14-19 years, and for university students, so... not only for beginners: Arzarello, Bazzini & Chiappini, 1994; as regards High School students, we consider fundamental: Matz, 1982). Why? Can we state that students simply *forget* that functions like $x\rightarrow x^2$, $x\rightarrow\sqrt{x}$, $x\rightarrow\sin x$, $x\rightarrow\log_e x$, ... cannot be considered as linear mappings?

We shall call the misconception that causes those mistakes *misconception of linear mappings*. But, as we shall see, this is just an operational misconception (caused by an over-use of metaphorical projections, too): it seems that several students do not really *think* that the mentioned functions are linear mappings; however, their behaviour is sometimes clear: operationally, $x \rightarrow x^2$, $x \rightarrow \sqrt{x}$, $x \rightarrow \sin x$, $x \rightarrow \log_e x$, ... are frequently considered as linear mappings (see: Markovitz, Eylon & Bruckheimer, 1986). So, according to A. Sfard, «there is probably much more to mathematics than just the rules of logic. It seems that to put out finger on the source of its ostensibly surprising difficulty, we must ask ourselves the most basic epistemological questions regarding the *nature of mathematical knowledge*» (Sfard, 1991, p. 2).

2. ANOTHER COMMON MISTAKE

Students know that:

$$\begin{aligned} A(x) = B(x) & \Leftrightarrow A(x) + c = B(x) + c & (c \in \mathbf{R}) \\ A(x) = B(x) & \Leftrightarrow k \cdot A(x) = k \cdot B(x) & (k \in \mathbf{R} \wedge k \neq 0) \end{aligned}$$

Sometimes, these rules are improperly extended and bring to the mistakes:

$$\begin{aligned} A(x) = B(x) & \Rightarrow \sqrt{A(x)} = \sqrt{B(x)} \\ [A(x)]^2 = [B(x)]^2 & \Rightarrow A(x) = B(x) \end{aligned}$$

and, with reference to inequalities:

$$A(x) < B(x) \Leftrightarrow k \cdot A(x) < k \cdot B(x) \quad (k \in \mathbf{R} \wedge k > 0)$$

We shall call this misconception *balance misconception* (it is not very different from the *misconception of linear mappings*, previously introduced: so it can be considered an operational misconception): if we “do” the same “operation” *op* in both members of the equation (inequality), we shall obtain a new equation (inequality) equivalent to the original one:

$$A(x) = B(x) \Rightarrow op[A(x)] = op[B(x)] \quad \forall op$$

But, of course, this is not always true... How does it happen that there are similar mistakes in protocols by High School pupils (regarding students aged 14-19 years)?

It is important to underline that the mistakes we mentioned can be considered similar; there is a correct rule, and it is rather “simple”: students accept it as a natural, reliable one. So they seem to be induced to extend (improperly) this rule to other cases, that are quite different from the original one.

In this paper, we shall analyse some improper generalisations of “simple rules”, with reference to High School students (students aged 16-19 years). We shall open our study with the investigation of some cases; we shall consider:

- a mistake about linear functions (the case of Sandra);
- a mistake about the solution of a quadratic equation (the case of Alberto);

- a mistake about Calculus (the case of Matteo).

Then we shall present:

- an experimental research about trigonometry.

3. SOME HIGH SCHOOL STUDENTS

3.1. Sandra and linear mappings

Let us consider the case of Sandra, an High School student of average mathematical skill, aged 17 years (4th class of a *Liceo scientifico*, in Treviso, Italy); she wrote in a protocol the following solution of the equation (being $x \in \mathbf{R}$):

$$\log_e(x^2+7) = \log_e 7 \Rightarrow \log_e x^2 + \log_e 7 = \log_e 7 \Rightarrow \log_e x^2 = 0 \Rightarrow x = 1$$

and (in the same protocol) the equality:

$$\sqrt{a^4 + 9 \log_e e + \log_e 1} = \sqrt{a^4 + 9 \cdot 1 + 0} = \sqrt{a^4 + 9} = a^2 + 3$$

So she considered the functions $x \rightarrow \log_e x$ and $x \rightarrow x^2$ as a linear mappings.

Sandra's interview took place in the classroom, in other pupils' presence:

Teacher: «First of all, let us consider the development of: $\log_e(x^2+7)$. Why did you write $\log_e(x^2+7) = \log_e x^2 + \log_e 7$?»

Sandra: «It seemed to me a natural thing to do».

Teacher: «Why?».

Sandra: «I think that there is a property that states something like this».

Teacher: «There are no properties to simplify $\log_e(a+b)$. Do you remember any property about it?»

Sandra: «I don't remember, now: we studied the function $x \rightarrow \log_e x$ several months ago».

Teacher: «So why did you think that $\log_e(a+b) = \log_e a + \log_e b$?»

Sandra: «Well, it seems correct».

[Let us underline that the statement «it seems correct» is really interesting: Sandra understood that her solution is incorrect, in teacher's opinion, but she noticed that *it seems* (at the moment of the interview) *correct*; she did not say *it seemed correct* (at the moment of the test): this suggests that, at the moment of the interview, she was still persuaded about the correctness of the 'property' $\log_e(a+b) = \log_e a + \log_e b$.

Teacher: «And is it correct?»

Sandra (worried): «No».

Teacher: «Why? Did you change your mind?»

Sandra: «You told me I made a mistake. So I made something wrong».

[So Sandra hardly accepts the teacher's correction. Then the teacher proposes a counterexample].

Teacher: «Let us consider this example: $\log_e(e+1)$. We know that $\log_e e = 1$ and $\log_e 1 = 0$. Well, if now we should accept that $\log_e(a+b) = \log_e a + \log_e b$, we must write: $\log_e(e+1) = \log_e e + \log_e 1 = 1+0 = 1$. Do you think it is true?»

Sandra (after a moment): «No. It is false».

Teacher: «Why?»

Sandra: «Of course it is false: 1 is $\log_e e$ so it cannot be $\log_e(e+1)$ ».

[So Sandra implicitly states that the function $x \rightarrow \log_e x$ is injective. But we are not sure that this statement is based upon a deep, conscious knowledge of that function: did Sandra extend injectivity from a (linear) function, like $x \rightarrow kx$ ($k \neq 0$), to the function $x \rightarrow \log_e x$?]

Teacher: «So $\log_e(a+b)$ is not $\log_e a + \log_e b$. Do you agree?».

Sandra (calm): «Yes, of course. But *it seems* correct, doesn't it?»

[Then Sandra wrote the correct solution of the equation:

$$\log_e(x^2+7) = \log_e 7 \Rightarrow x^2+7 = 7 \Rightarrow x^2 = 0 \Rightarrow x = 0]$$

Teacher: «The solution is correct. Now let us see: $\sqrt{a^4+9}$. Why did you write $\sqrt{a^4+9} = a^2+3$?»

Sandra: «I thought: $\sqrt{a^4} = a^2$ and $\sqrt{9} = 3$ ».

Teacher: «Do you think your process is correct?»

Sandra: «Well, $\sqrt{a^4}$ is really a^2 and $\sqrt{9}$ is 3».

Teacher: «Of course: but is it correct to say that $\sqrt{a^4+9} = a^2+3$?»

Sandra (smiling): «I guess the answer is no, isn't it? But how can I simplify $\sqrt{a^4+9}$?»

Teacher: «There are no properties to simplify $\sqrt{a^4+9}$ »

Sandra: «No properties about $\log_e(a+b)$, no properties about $\sqrt{a^4+9}$. But it was possible to solve the equation: and now, how can I simplify $\sqrt{a^4+9}$?»

Teacher: «You cannot simplify it».

Sandra: «So when I find $\sqrt{a^4+9}$ I must stop».

Teacher: «Of course: you cannot continue: $\sqrt{a^4+9}$ is the final result of your exercise».

Sandra (worried): «Agreed».

Sandra does not seem quite persuaded: she explicitly noticed that the equation $\log_e(x^2+7) = \log_e 7$ can be solved without writing $\log_e(x^2+7) = \log_e x^2 + \log_e 7$; so, in this case, *there is* a correct route to be taken, instead of the wrong route. The case $\sqrt{a^4+9}$

seems a different one; Sandra asked *twice*: «how can I simplify $\sqrt{a^4+9}$?» It is impossible to «continue» this exercise without writing $\sqrt{a^4+9} = a^2+3\dots$

It is very interesting to consider the final part of the interview:

Teacher: «For example, do you think that $e^{a+b} = e^a+e^b$?»

Sandra (lit up with joy!): «Oh no, no! I know very well that $e^{a+b} = e^a \cdot e^b$!»

So Sandra will not fall in mistakes like $e^{a\pm b} = e^a \pm e^b$: in these cases *she knows* some simple rules to simplify $e^{a\pm b}$ ($e^{a+b} = e^a \cdot e^b$ and $e^{a-b} = e^a/e^b$), so she is not forced to extend (improperly) other rules.

However, let us underline that this situation is not referred to an exercise (so it is not influenced by the didactic contract): the last answer can be influenced by the experimental contract (see for instance: Schubauer Leoni, 1988; Schubauer Leoni & Ntamakiliro, 1994).

3.2. Alberto and equations

Let us consider the case of Alberto, an High School student of average mathematical skill, aged 16 years (3rd class of *Liceo scientifico*, in Treviso, Italy); he wrote in a protocol the following solution (being $x \in \mathbf{R}$):

$$5x^2 = 20 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

Of course, the teacher underlined that this solution is wrong because it “forgets” the root $x = -2$. He showed the following (correct) solution:

$$5x^2 = 20 \Rightarrow x^2 = 4 \Rightarrow x^2 - 4 = 0 \Rightarrow (x+2)(x-2) = 0 \Rightarrow x = -2 \vee x = 2$$

Alberto’s interview took place in the classroom, in other pupils’ presence:

Teacher: «What about this solution?»

Alberto: «Of course it is correct, but it is rather strange, difficult for me. I did not think that the rule $a^2-b^2 = (a+b)(a-b)$ is necessary to solve an equation».

Teacher: «It is not absolutely necessary: it is enough to remember that $2^2 = (-2)^2 = 4$. Please, describe entirely your solution».

Alberto: «First of all, I divided both members by 5; then I calculated the square roots of both members: this is easy. And it is very easy to be kept in mind...»

So Alberto states that he chose the simplest route, the solution «easy to be kept in mind»; he «did» the same «operations» in both members of the equation, hoping to obtain a new equation equivalent to the original one: the mistake seems clear, now.

Well, is everything clear, after the mentioned interview?

In another protocol, *just a month later*, Alberto wrote:

$$y^2 - x^2 = 2x + 1 \Rightarrow y^2 = x^2 + 2x + 1 \Rightarrow \sqrt{y^2} = \sqrt{(x+1)^2} \Rightarrow y = x + 1$$

So we can point out that the *balance misconception* is very strong, lasting: even the teacher’s correction was not effective enough to overcome it, in Alberto’s mind.

3.3. Another example

We underline once again that the mistakes previously considered can be found in several school levels: so they are not made only by beginners. Let us briefly see the case of Matteo, an High School student aged 18 years (5th class of *Liceo scientifico*, in Treviso, Italy): in a protocol, he wrote down the following correct process:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 3x + 2} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} = \lim_{x \rightarrow 1} \frac{(x+1)}{(x-2)} = -2$$

that can be represented as:

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow +\infty} \frac{f(x)/h(x)}{g(x)/h(x)}$$

But in the same protocol, just few rows later, Matteo... extended it to the following wrong process:

$$\lim_{x \rightarrow +\infty} \frac{\sqrt{x}}{x+1} = \lim_{x \rightarrow +\infty} \frac{(\sqrt{x})^2}{(x+1)^2} = \lim_{x \rightarrow +\infty} \frac{x}{x^2 + 2x + 1} = 0$$

(where... only the final result is correct!) that can be represented as:

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow +\infty} \frac{[f(x)]^2}{[g(x)]^2}$$

Can we consider it only a “casual” mistake?

In our opinion, in this situation we can underline the clear influence of the (correct) habit to simplify some fractions in order to calculate a limit; the student tried to extend (incorrectly) this process, so we can point out the presence of a misconception similar to the *balance misconception* (applied to fractions).

We shall not examine closely Matteo’s mistake: we wanted just underline that some mistakes that can be connected to the *balance misconception* are present in students aged 18-19 years (5th class of Italian *Liceo scientifico*; as regards the learning of the notion of limit, and in particular some important misconceptions, see for example: Cornu, 1980; Davis & Vinner, 1986; Dimarakis & Gagatsis, 1996).

4. EDUCATIONAL ROOTS OF ALGEBRAIC THOUGHT: FROM ARITHMETICS TO ALGEBRA

We pointed out that students sometimes improperly extend «simple» rules. Sandra (paragraph 3.1) seemed to look for any rule to simplify some expressions, so she considered the functions $x \rightarrow \log_e x$ and $x \rightarrow x^2$ as linear mappings; but let us remember the final part of her interview, too: it seems that mistakes like $e^{a \pm b} = e^a \pm e^b$ are not very frequent, because in these cases *students know* some rules to simplify $e^{a \pm b}$ ($e^{a+b} = e^a \cdot e^b$, $e^{a-b} = \frac{e^a}{e^b}$).

and $e^{a-b} = e^a/e^b$): so they are not forced to extend improperly other rules (we shall deal with this situation in the following paragraph).

Nevertheless, the situation is not simple: for example, we cannot forget the mistake (that can be noticed in pupils, aged 18-19 years, that are studying Calculus) by which the derivative of the product $f(x) \cdot g(x)$ would be the product of the derivatives of the respective factors, $f'(x) \cdot g'(x)$: it seems connected with the (correct) rule by which the derivative of the sum $f(x)+g(x)$ is the sum of the derivatives of the respective addends, $f'(x)+g'(x)$. So sometimes students improperly extend a simple and well-known rule also when there is a rule by which it is possible to solve the problem in question (the derivative of a product of functions).

From a technical point of view, we should say that the examined mistakes are based upon algebraic weakness (let us consider for example Alberto's mistake and the *balance misconception*, paragraph 3.2, or Matteo's mistake, paragraph 3.3; let us quote once again: Tietze, 1988; Malle, 1993). It is interesting to examine briefly some educational roots of algebraic thought.

F. Arzarello, L. Bazzini and G. Chiappini notice: «Several Authors state that the roots of algebraic thought can be pointed out in the effort to consider a computational process in a quite general way» (Arzarello, Bazzini & Chiappini, 1994, p. 10). So Algebra itself is often introduced as a generalisation: for example, we generalise arithmetical operations by some algebraic process; and this «replacement» of Arithmetics by Algebra is sometimes a source of obstacles. Y. Chevallard writes: «For several generations, Arithmetics was the green Paradise of [...] the spirit opening to an marvellous intellectual activity [...] So an only too well learnt Arithmetics became an intellectual, affective and ideological obstacle against its overcoming» (Chevallard, 1989, p. 15; see moreover: Vergnaud, Cortes & Favre-Ortigue, 1997, p. 253).

Moreover, let us underline that the role of formal transformations is important. L. Bazzini notices: «Students' answers to questions about equivalence of equations (or inequalities) are highly influenced by presence or absence of formal transformations. This [...] entails a careful reflection about cognitive processes in learning of Algebra» (Bazzini, 1995, p. 44; see moreover: Linchevski & Sfard, 1991; Sfard & Linchevski, 1992; Arcavi, 1994; Cortés, 1994).

It is important that the students (and the High School students, too) are enabled to interpret algebraic symbols and processes *not* only from a syntactic point of view (Burton, 1988; Tall, 1990).

So we can resume:

- in the examined students' mistakes, we can point out obstacles related to algebraic weakness: so some pupils do not aware several basic algebraic techniques;
- moreover, we must underline the presence of obstacles related to affective sphere: students know a "simple" rule, they use successfully in many cases, so associate those rules to good performances; when they have no rules to use in some resolutions, they improperly extend those rules, hoping to have, once again, good performances.

Then we wanted to examine the fundamental affective roots of the obstacles that may cause some of the mistakes previously examined (according to: Chevallard, 1989).

By the following tests we wanted to point out that affective aspect is fundamental to settle this situation.

5. AN EXPERIMENTAL RESEARCH

5.1. Method of tests

Two tests were proposed to students belonging to two 4th classes and two 5th classes of a *Liceo scientifico* (High School; pupils aged 17-19 years) in Treviso, Italy, total 95 students (two 4th classes: 23 and 24 pupils respectively; two 5th classes: 26 and 22 pupils respectively); we shall identify them by group A (the first 4th and 5th classes, total 49 pupils) and group B (the second 4th and 5th classes, total 46 pupils; all students had the same mathematics teacher; their curricula were standard: they knew basic elements of trigonometry; in particular, they knew the fundamental equality: $\sin^2x + \cos^2x = 1$).

The first test (A) was proposed to the 49 pupils of the group A (4th class: 23 pupils; 5th class: 26 pupils):

A) You know that $\sin^2x + \cos^2x = 1$ is true for every x . Is the equality:
 $\sin^4x + \cos^4x = 1$
 true for every x ?

Time: 1 minute (we wanted that students examine the problem ‘at a glance’). No textbooks or electronic calculators allowed.

The second test (B) was proposed to the 46 pupils of the group B (4th class: 24 pupils; 5th class: 22 pupils):

B) Let $a^2 + b^2 = 1$. Is the equality:
 $a^4 + b^4 = 1$
 true for every a and for every b such that $a^2 + b^2 = 1$?

Time: 1 minute. No textbooks or electronic calculators allowed.

By these tests we wanted to examine the influence of the well-known trigonometric rule $\sin^2x + \cos^2x = 1$ in the interpretation of the (incorrect) equality $\sin^4x + \cos^4x = 1$ (test A). The test B is based upon the incorrect statement $a^2 + b^2 = 1 \Rightarrow a^4 + b^4 = 1$, that can be considered technically equivalent to the problem expressed in the test A, but it has no reference with the basic rule $\sin^2x + \cos^2x = 1$.

5.2. Results of the tests

Group A						
	4 th class (23 students)		5 th class (26 students)		Total (49 students)	
Yes (true)	9	39%	12	46%	21	43%
No (false)	10	44%	8	31%	18	37%
No answer	4	17%	6	23%	10	20%

<i>Group B</i>						
	4 th class (24 students)		5 th class (22 students)		Total (46 students)	
Yes (true)	4	17%	2	9%	6	13%
No (false)	19	79%	16	73%	35	76%
No answer	1	4%	4	18%	5	11%

5.3. Considerations about results

These results show a clear difference: as regards the group A, 43% of the students stated that $\sin^4x + \cos^4x = 1$; probably, they were influenced by the (correct) rule $\sin^2x + \cos^2x = 1$. As regards the group B, in fact, only 13% of the students stated that $a^2 + b^2 = 1 \Rightarrow a^4 + b^4 = 1$.

Let us remember that several students (19 out of 35 students that answered *no* or *false* to the question of the test B) noticed that $a^2 + b^2 = 1 \Rightarrow a^4 + b^4 = 1$ can be true only for some particular values of a and b ($a = \pm 1$ and $b = 0$, or $a = 0$ and $b = \pm 1$).

5.4. Justifications given by students

Several students gave interesting justifications; as regards the students that answered *yes* or *true* to the question of the test A, let us remember the following justification:

«I saw $\sin^4x + \cos^4x = 1$ and I immediately thought that $\sin^4x = (\sin^2x)^2$ and $\cos^4x = (\cos^2x)^2$, so I concluded that $\sin^4x + \cos^4x = 1$ is true» (Aldo, 4th class); 9 justifications (test A) and 4 justifications (test B) are similar to this one.

In Aldo's justification we can clearly point out the presence of the *misconception of linear mappings*.

«Of course, I realise I've made a big mistake, I do not know the reason: I remembered the famous trigonometric rule $\sin^2x + \cos^2x = 1$ and I thought that it could be true for 4, too. But why? I did not remember any other rules about $\sin^4x + \cos^4x = 1$ so I tried to apply the one and only rule I could remember» (Anna, 5th class).

Anna's justification is not very different from Sandra's one (see paragraph 3.1): she had no particular rules for $\sin^4x + \cos^4x = 1$, so she extended a well-known rule...

As regards students that answered *no* or *false* to the question of the test B, let us remember:

«The equality is false: a^4 and b^4 are not equal to a^2 and b^2 : it is true only for some particular cases» (Antonio, 5th class).

As previously remembered, 19 out of 35 students that answered *no* or *false* to the question of the test B underlined that $a^2 + b^2 = 1 \Rightarrow a^4 + b^4 = 1$ can be true only for some particular values of a and b .

We can conclude that the presence itself of a 'rule' just similar to the 'famous' formula $\sin^2x + \cos^2x = 1$ induced many students to refer to it: in the test A the well-

known rule is explicitly present; on the contrary, in the test B the (incorrect) statement $a^2+b^2 = 1 \Rightarrow a^4+b^4 = 1$ is not referred to any ‘famous’ rule.

6. CONCLUSIONS

6.1. Affective roots of some misconceptions

Results of the experimental research previously presented needs some remarks.

First of all, let us remember that several researches showed that the different representations of a problem are very important as regards students’ behaviour in problem solving (see: Fischbein, Tirosh & Hess, 1979; Silver, 1986; Arcavi, Tirosh & Nachmias, 1989; see moreover: Gagatsis & Thomaidis, 1995).

The situations previously described show that a ‘simple’ rule is often seen as a natural and a reassuring one. So, from an affective point of view, too, some students are induced to apply it to a lot of cases, without particular controls: of course, this can cause dangerous mistakes.

Many students try to extend a well-known rule when they do not know specific rules to solve a problem. Let us remember, for example, that it is well known that pupils are afraid of problems ‘without a result’ (about ‘impossible’ problems, see: Baruk, 1985; Micol, 1991; Schubauer Leoni & Ntamakiliro, 1994; a systematic classification of ‘impossible’ problems can be found in: D’Amore & Sandri, 1993). So we can state that this fear brings many students to solve the considered problem by a familiar, reliable rule: unfortunately sometimes this rule cannot be applied to the considered case. (Let us notice that in a recent work, B. D’Amore and P. Sandri studied some situations related to problems with a missing datum, with reference to pupils aged 8-9 years and 12-13 years: they noticed that many pupils ‘imagine’ the missing datum in order to be able to solve the problem: D’Amore & Sandri, 1998; they refer to Brousseau, 1986; we could say that an almost similar behaviour can be pointed out in High School students, too: when they feel they cannot solve *completely* a problem, for example because there are no rules to simplify a formula, they try to apply a well-known rule in order to solve completely the problem in question: this can be sometimes reassuring, but it is wrong).

6.2. How can we overcome these misconceptions?

As we shall see, it is not easy to overcome these misconceptions. We should say that the role of counterexamples is important to make students aware of incorrect answers and of their conflicting ideas.

For example, as regards the mistake $\sin^4x+\cos^4x = 1$ (for every $x \in \mathbf{R}$, paragraph 4.1, test A), it is easy to show directly that if we consider the case $x = \frac{\pi}{4}$, we have:

$$\sin^4\left(\frac{\pi}{4}\right) + \cos^4\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^4 + \left(\frac{\sqrt{2}}{2}\right)^4 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 1$$

Moreover, if we consider the *misconception of linear mappings*, we can remember the Theorem of Pythagoras (a, b, c are measures of three sides of a triangle having a right angle, and a is hypotenuse's measure), by which we can write:

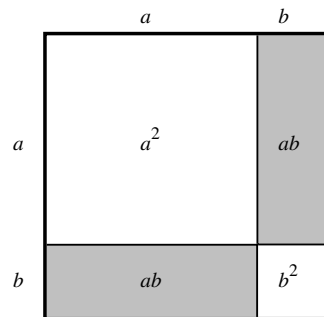
$$a = \sqrt{b^2 + c^2}$$

Of course, if we write:

$$a = \sqrt{b^2 + c^2} = \sqrt{b^2} + \sqrt{c^2} = b+c$$

we should state that the sum of two sides of a triangle is equal to the third one, and this is clearly an absurd statement.

As regards the role of counterexamples, visualisation can be very important. Let us remember that some Authors, in the last years, worked about matters connected to visualisation. R. Duval notices that «mathematical objects are not directly accessible to the perception [...] as objects generally said 'real' or 'physical'; so he states that «different semiotic representations of a mathematical object are absolutely necessary» (Duval, 1993, p. 37). The important presence of different registers of representation is, in Duval's opinion, quite remarkable: «The cognitive functioning of human thought is inseparable from the existence of a variety of semiotic registers of representation. If we call *sémiosis* the learning of the production of a semiotic representation and *noésis* the conceptual learning of an object, we must affirm that *sémiosis* is inseparable from *noésis*» (Duval, 1993, pp. 39-40). A well-known work by E. Fischbein is devoted to visual representation of mathematical objects and to its great importance in mathematics education; Fischbein states that «the integration of conceptual and figural properties in unitary mental structures, with the predominance of the conceptual constraints over the figural ones, is not a natural process. It should constitute a continuous, systematic and main preoccupation of the teacher» (Fischbein, 1993, p. 156; as regards functions, see moreover: Vinner, 1983, 1987 and 1992).



The following example is referred to the 4th Proposition of Euclidean Geometric Algebra (that is: «If a segment is divided, the area of the square of the whole segment is equal to the sum of the areas of the squares of the two parts and of the double of the area

of the rectangle contained between such parts»: Euclid, 1970, p. 163 (as regards Euclidean Geometric Algebra from historical point of view, see for example: Boyer, 1968; Kline, 1972; van der Waerden, 1983; Anglin, 1994).

Nowadays the 4th Proposition of Euclidean Geometric Algebra can be expressed by:

$$(a+b)^2 = a^2+b^2+2ab$$

but in *Elements* only the previous picture gives the proof of this proposition.

It is clear that the mistake that identifies improperly $(a+b)^2$ in a^2+b^2 (without the so-called “double product”, according to the *misconception of linear mappings*) is nearly impossible if the visual representation is correctly considered (see for example: Kaldrimidou, 1987; Bagni, 1997).

Nevertheless, we cannot say that the use of counterexamples is always conclusive: *the effect of counterexamples with students is often weak since they are not able to interpret given counterexamples in an adequate way.*

As regards this important point, let us remember once again Alberto’s case; the pupil wrote: $x^2 = 4 \Rightarrow x = 2$. In order to correct this mistake, the teacher underlined that: $2^2 = (-2)^2 = 4$; but clearly this correction was not effective enough to overcome completely the *balance misconception*, in Alberto’s mind. In fact the pupil simply said: «I calculated the square roots of both members». So Alberto understood teacher’s statement, but he was not able to interpret teacher’s correction in the sense of a real counterexample, strictly related to his previous mistake: he was not able to connect directly and effectively $2^2 = (-2)^2 = 4$ (square powers) to the correct solution, $x^2 = 4 \Rightarrow x = -2 \vee x = 2$ (square roots).

Then it seems that some misconceptions are really lasting: although they can be sources of inconsistencies in student’s minds, they reoccur and their effects can be pointed out several times (as regards the presence of conflicting answers and of ideas that are incompatible with each other, see for example: Tall, 1990; Tsamir & Tirosh, 1992; let us remember that several researches showed that sometimes students do not realise the presence of conflicting answers: Stavy & Berkovitz, 1980; Hart, 1981: for example, the persistence of different sorts of algebra errors in pupils aged 11-18 years is clearly proved: Matz, 1982; and sometimes the presence of ideas that are incompatible with each other is not considered completely illicit, forbidden: Schoenfeld, 1985; Tirosh, 1990).

6.3. General conclusions

We do not think that the obstacles previously examined can be considered as epistemological ones or (only) as educational ones (see for example the fundamental classification in: Brousseau, 1983; Vergnaud, 1989, pp. 168-169). If we consider them as educational obstacles, we must underline that the influence of affective aspect is surely remarkable. Then, in our opinion, they can be regarded as *affective obstacles*, too: so it is difficult to overcome them completely just by educational means, like for example showing of counterexamples (D’Amore & Martini, 1997).

Of course, we must underline that analogical reasoning should not be too quickly dismissed: in fact, many mathematicians used and use it as one of the main ways for creating new mathematics! However, the really different propensity for self-correction

should be considered, when we compare research mathematicians and young students: for example, frequently mathematicians employ analogical reasoning in formulation of a conjecture, whose logical soundness must be deeply verified; on the other hand, generally students do not perform this meta-discursive monitoring.

We underlined that the *misconception of linear mappings* and the *balance misconception* are operational misconceptions, caused by an improper over-use of metaphorical projections, too: this suggests some remarks.

It is clear that «in order to speak about mathematical *objects*, we must be able to deal with *products* of some processes without bothering about the processes themselves [...] It seems, therefore, that the structural approach should be regarded as the more advanced stage of concept development. In other words, we have good reasons to expect that *in the process of concept formation, operational conceptions would precede the structural*» (Sfard, 1991, p. 10). If we accept, according to A. Sfard, that concept formation takes place by «a *hierarchy*, which implies that one stage cannot be reached before all the former steps are taken», we must conclude that the full development of these phases (A. Sfard calls the stages in concept development «*interiorization, condensation and reification* respectively»: Sfard, 1991, p. 18) is not a natural process, but it must be carefully controlled by the teacher.

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