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Georg Pick's reticular Geometry and Didactics of Mathematics

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Summary. In this paper the original article (Prague, 1899) by Georg Pick about some questions in reticular geometry is examined. The didactical importance of this subject in High School (referred to 4th class of Italian *Liceo scientifico*, pupils aged 17 years) is studied by a test.

INTRODUCTION

In the last years, several works were published about reticular geometry (see for example [Stocco, 1986] [Gambarelli, 1989] [Bagni, 1990]), from the didactical point of view, too (see [Kaldrimidou, 1995]). This particular interest makes it advisable an historical and didactical reprise of Pick's plane (the Euclidean plane with the straight lines that, referred to a Cartesian system, are represented by the equations $x = m$, $y = n$, being $m\in\mathbf{Z}$, $n\in\mathbf{Z}$).

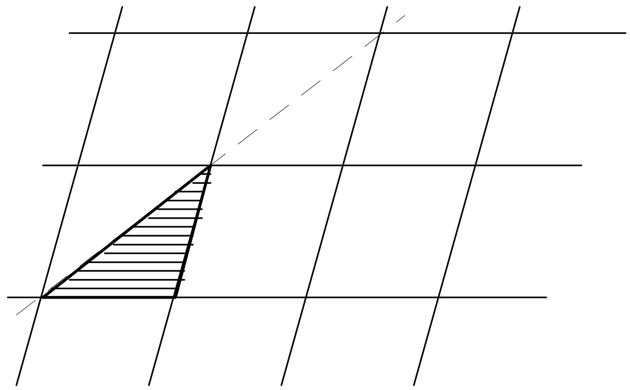
In this work we shall include a short presentation of the original work by Georg Pick [Pick, 1899]; then we shall examine the introduction of the main concepts by Pick in High School (pupils aged 17-18 years).

1. THE ORIGINAL WORK BY GEORG PICK (1899)

Pick's original work about reticular geometry is entitled *Geometriches zur Zahlenlehre* (*Geometry for Number Theory*); it is the text of a relation to the German Mathematical Society of Prag and it was published in 1899 [Pick, 1899].

Pick introduces "two systems of parallel straight lines in the plane, having the same distance", named *main reticular lines*; the intersections of these lines are named *reticular points* [Pick, 1899, p. 311]; the straight lines for more than a reticular point are named *reticular lines*. The surface measure is "the half of a single mesh" [Pick, 1899, p. 312].

A polygon whose vertexes are reticular points is named *reticular polygon*. So all sides of a reticular polygon belong to reticular lines [Pick, 1899, p. 312].



Pick divides a reticular polygon into two polygons by a reticular line for two reticular points belonging to its perimeter. Let i be the number of the reticular points inside, u the number of the reticular points belonging to its perimeter, i_1, u_1, i_2, u_2 the numbers of the reticular points of the new polygons so obtained; let δ be the number of the reticular points belonging to the part of the reticular line that divides the polygon into the two parts.

So we have:

$$i = i_1+i_2+\delta \qquad\qquad u = u_1+u_2-2\delta-2$$

It follows that:

$$2i+u-2 = (2i_1+u_1-2) + (2i_2+u_2-2)$$

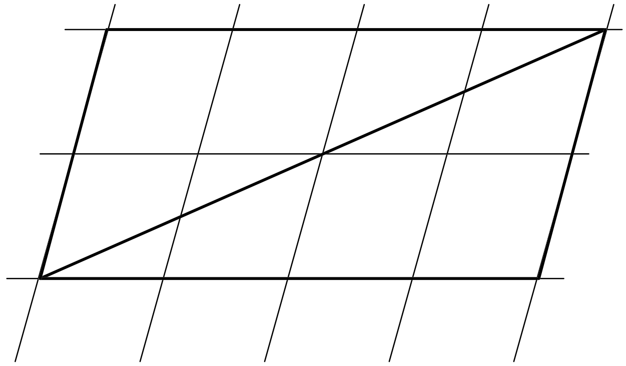
Pick denotes $(2i+u-2)$ by *number of points* of the considered polygon [Pick, 1899, pp. 312-313]; "the number of points of a polygon made by two parts is the sum of the numbers of points of the parts" [Pick, 1899, p. 313].

The number of points of a reticular polygon is very important: Pick states that "for every reticular polygon, the area is its number of points" [Pick, 1899, p. 314] (the area is calculated with respect to the measure previously chosen, that is the half of a single mesh).

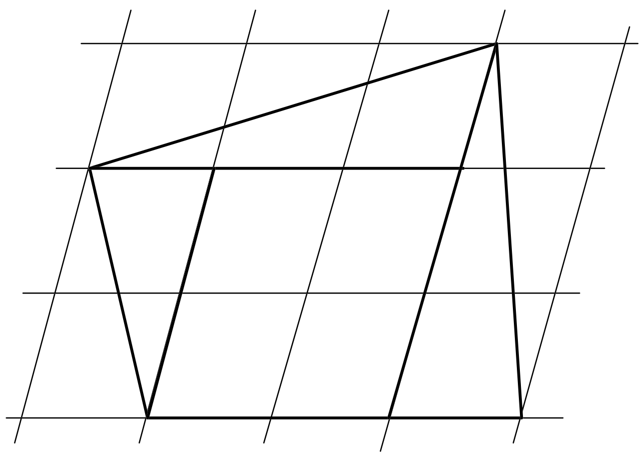
To prove that, Pick notes that this property is true for a polygon made by a single mesh:

$$i = 0 \ \wedge \ u = 4 \qquad \text{so:} \qquad 2i+u-2 = 2$$

This property is true for a polygon having all its sides belonging to main reticular lines, too, "from the property of composition" [Pick, 1899, p. 313].



Moreover, if we subdivide a parallelogram whose perimeter belongs to main reticluar lines into two equal triangles, having a diagonal in common, the number of points of one of them is the half of the number of points of the parallelogram; so the number of points is its area.



Pick notes that any reticular polygon can be subdivided into parallelograms whose perimeters belong to main reticular lines and in triangles obtained by halving these parallelograms by a diagonal. By that, it follows that the area of every reticular polygon is its number of points [Pick, 1899, p. 314].

2. PICK'S PLANE: A DIDACTICAL EXPERIENCE

2.1. THE AREA OF A RETICULAR POLYGON

As above seen, the main result of reticular geometry is the formula:

Area of a reticular polygon (referred to the area of a singhe mesh) =

$$= \frac{1}{2}\cdot(2i+u-2) = i+\frac{u}{2}-1$$

Some students of High School were asked to deduce this formula.

2.2. METHOD OF OUR RESEARCH

The following test was proposed to students belonging to a 4^{th} class of a *Liceo scientifico* (High School) in Treviso, Italy (pupils aged 17-18 years), total 24 students (their mathematical curricula were standard; they knew the words: *mesh*; *reticular point*; *reticular polygon*; they considered square meshes):

> *Find a formula to calculate the area of a reticular polygon (referred to the area of a single mesh), based upon the number i of the internal reticular points of the polygon and upon the number u of the reticular points belonging to its perimeter.*

Time: 30 minutes.

2.3. RESULTS OF THE TEST AND CONSIDERATIONS

Only 4 students out of 24 (Andrea, Guido, Martino and Nicoletta) obtained the correct formula. Other students made only attempts, with the consideration of some particular figures.

As we shall see, the students that found the correct formula did not give a real, complete proof of it; in particular, they considered some particular cases, from that they obtained the formula.

2.4. JUSTIFICATIONS GIVEN BY STUDENTS

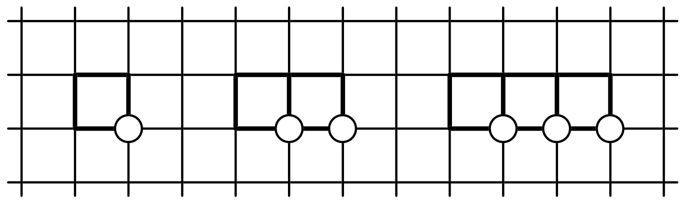
The students were asked to justify their statements. The greater part of the students that did not find the correct formula admited that they considered only (many) particular figures.

2.4.1. Andrea

Andrea: ‘I thought that the formula was... a formula of degree 1”.

Interviewer: ‘Why?”

Andrea: "Of course, I began from the lowest degree. Moreover, if we associate a single mesh to every point, for example the upper left mesh, we understand that if the number of the points increases, the area increases, too. Of course the easy formula A = $u+i$ is quite wrong".

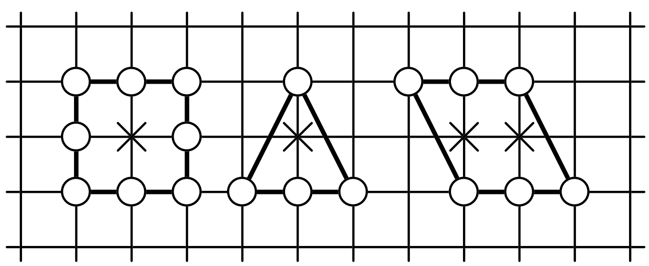


(Andrea draws the rectangles).

Andrea: "When I understood that A = $u+i$ was not the correct formula, I looked for another one, always a formula of degree 1. Something like:

$$A = a \cdot u + b \cdot i + c$$

with a, b, c numbers. I tried to obtain a, b, c by a system. I considered three easy figures".



(Andrea draws the figures).

Interviewer: "Why did you consider those figures?"

Andrea: "I began with easy figures having some internal points. The small square [the square made just by a single mesh], for example, has no internal points and I do not know if it can be useful. From that, I found:

$$\begin{cases} 8a + b + c = 4 \\ 4a + b + c = 2 \\ 6a + 2b + c = 4 \end{cases}$$

I solved the system and I found:

$$a = \frac{1}{2}; \qquad b = 1; \qquad c = -1$$

So the formula is:

$$\mathrm{A} = \frac{1}{2}u + i - 1$$

Then I made some controls and I saw that this formula is correct".

Interviewer: "Why did you t hink that further checks were necessary?"

Andrea: "Well, I wanted to control my calculations".

Interviewer: "Apart from calculations, were you quite sure about your method?"

Andrea: "Yes, of course. It is just a system. I can solve a syestem".

Interviewer: "You supposed that the formula was a formula of degree 1 in the variables *u* and *i*. You cannot be sure about that".

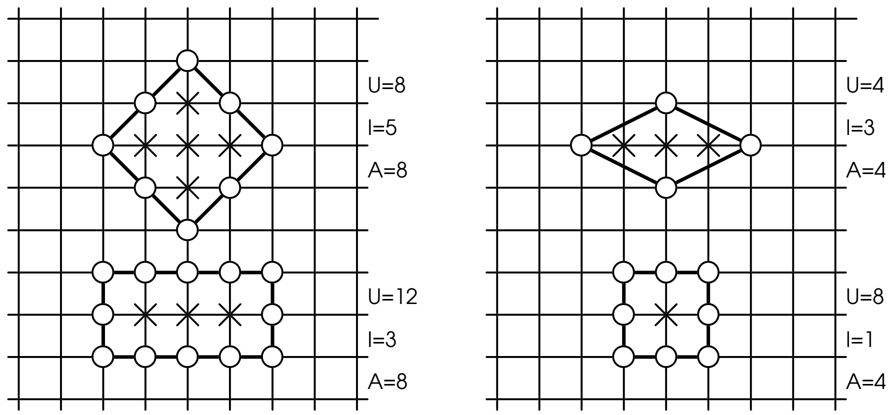
Andrea (smiles): "Alright, but I knew that the formula was based upon *u* and *i*, and I guessed it was a formula of degree 1. And then... my formula is right, isn't it?".

We shall analyze Andrea's statements in the following paragraph; the most interesting parts of his method are based upon a direct attempt to obtain the formula, without particular figures; first of all, he supposed that the formula was a formula of degree 1 (and this is very important), then he considered some polygons to obtain its coefficients.

2.4.2. Martino

Martino: "I worked on two couples of figures, having the same area but having different numbers of reticular points on their perimeters and different numbers of internal points. At the beginning, I did not know what to do, so I made some attempts".

Interviewer: "Why did you draw two couples of polygons having the same area?"

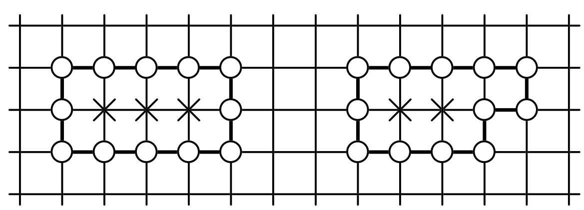


(Martino draws the figures and writes the numbers).

Martino: ‘I wanted to avoid particular cases, I did not want a particular formula”.

Interviewer: “And then?”

Martino: ‘I noted that the numbers of reticular points on the perimeters are, in this cases, greater than the areas. So I thought that this numbers may be halved. Moreover I thought that, for example in the rectangle with only a row of internal points, two points on the perimeter can be associated to every internal points. So I thought that halving the number of reticular points belonging to perimeter was probably a good idea. Then I considered something more”.



(Martino draws the figures).

Martino: ‘From these figures, I guessed that the area depends directly on the number of internal points: if the number of internal points decreases, leaving the same number of the reticular points belonging to the perimeter, the area decreases, too. So I thought: perhaps the area depends on the internal points, all of them, and on the half of the points belonging to the perimeter of the polygon. I made some checks, and I always obtained the values that I was waiting for, plus 1. So the correct formula would be $i+\frac{u}{2}-1$. Finally I checked this formula with many other figures: it seemed really correct".

We shall analyze Martino's statements, too, in the following paragraph; the most interesting parts of his method are based upon the consideration of polygons having the same area, by which he studied the different possibilities of internal reticular point and of reticular points belonging to the perimeter referred to a fixed value of the area. As we shall see, his care to avoid particular cases is interesting.

2.4.3. Guido and Nicoletta

Other two students that found the formula stated they took into consideration a great number of figures. In particular, Guido examined some polygons (rectangles and triangles) with no internal points:

Guido: "I considered many polygons and I noted that everyone of them can be divided into triangles and into rectangles. So I thought that these polygons are important. I examined rectangles and triangles with no internal points and I noted that their areas was always the half of the number of the points belonging to the perimeter minus 1. Then I considered rectangles and triangles having some internal points and I understood that I must add the number of internal points".

Nicoletta: "I studied many cases and I made a table with the numbers of the points belonging to the perimeter, with the number of the internal points and with areas. I saw that the half of the first number added to the second numbers was the third number plus 1, so I understood the formula".

2.5. CONCLUSIONS

Let us resume some interesting remarks about the justifications given by students:

- Only Andrea was looking for the formula without considering particular cases. Many other students began their works considering particular figures and polygons.
- Andrea's supposition about formula's degree is remarkable. He wanted to begin his work from a simple possibility. Later he explained this choice by some geometrical notes: perhaps some clauses of the *didactical contract* forced Andrea to give a formal justification [D'Amore, 1993].
- Then Andrea's method is clear. He knew that the formula depends upon the number u of the reticular points belonging to the perimeter and upon the

number i of the internal reticular points, so he considered "easy figures having some internal points".

- Martino's method is based upon the consideration of polygons having the same area to study the different possibilities about numbers of reticular points belonging to the perimeter and of internal reticular points; his worry to avoid particular cases can be caused by *didactical contract*. However, Martino's method is not based upon a "classical" proof: he knew by intuition the formula from the consideration of some important cases.
- Guido's statement is very interesting; he wanted to justify his choice to consider rectangles and triangles and he noted that (with reference to the considered polygon) "all polygons can be divided into rectangles and triangles"; but this property was not used by Guido himself to obtain the formula (he analyzed just some polygons... without internal points); however he was forced (perhaps by some clauses of the *didactical contract*) to justify his own choice [Brousseau, 1987].

We can finally note that reticular geometry is important in didactics of mathematics.

The use of uncommon methods is really interesting: employed methods are clearly far from absolutely rigorous processes, but they are didactically very important. Students were induced to solve the problem after that they considered several possibilities: and the choice of these possibilities is an important act. So methods used by several students (sometimes successfully) were not based just upon casual attempts, but they were referred to geometric matters, to clear and interesting considerations.

Moreover, let us note that the visualization (a very important, main question for the didactics of mathematics, see for example [Schoenfeld, 1986], [Duval, 1993] and [Fischbein, 1993]) strongly supports this way of doing. The visual kind of this problem implies that it is possible and often useful to realize easy attempts, visual considerations of some particular cases, to be generalized later in conclusions.

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