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An investigation of some misconceptions in High School students' mistakes

GIORGIO T. BAGNI

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ROMA "LA SAPIENZA", ITALY

Summary. In this paper we analyse some common mistakes, referred to linear mappings and to the solution of some algebraic equations (with reference to students aged 16-19 years). We examine some case studies and propose a brief experimental research: we conclude that pupils sometimes improperly extend algebraic rules, and this is caused by algebraic weakness and by relevant affective elements, too. As regards strategies to overcome misguided generalisations, we notice that the effect of counterexamples with pupils is frequently remarkably weak: sometimes they are not able to interpret correctly the proposed counterexamples. An institutionalization stage is needed in order to make counterexamples effective.

1. SOME COMMON MISTAKES

Looking for analogies and differences in different situations is educationally (and, in general, mathematically) interesting: abstraction itself is based upon interpretation of analogies and upon consideration of similar problems in different contexts. However it is necessary to consider and to prevent the possibility of mistakes in using analogy: for instance misguided generalisations can be frequently pointed out in students' behaviour. In this paper we shall investigate some common mistakes and their educational (and, as we shall see, affective) roots.

Let us quote A. Sfard:

> «The student may manipulate a concept through a certain prototype; for example, the data collected by Markovitz et Al., 1986, show that beginners tend to imagine linear mappings whenever the notion of function is mentioned» (Sfard, 1991, p. 21).

As a matter of fact, the property of a function to be a linear mapping seems to be a general rule for some students. Mistakes like:

Note. Some results proposed in the present work were published in: Bagni, 2000.

$(a+b)^2 = a^2+b^2$
$\sin(a+b) = \sin a+\sin b$
$\log_e(a+b) = \log_e a+\log_e b$

can be pointed out in students' protocols in several school-levels (Tietze, 1988 and Malle, 1993; for High School students, pupils aged 14-19 years, and for university students: Arzarello, Bazzini & Chiappini, 1994; as regards High School students, we consider fundamental: Matz, 1982).

We call the misconception causing those mistakes *misconception of linear mappings*: it seems that several students do not really think that the mentioned functions are linear mappings; however, operationally, $x \to x^2$, $x \to \sin x$, $x \to \log_e x$, ... are frequently considered as linear mappings (Markovitz, Eylon & Bruckheimer, 1986).

Let's consider another common mistake. Students know that:

$$A(x) = B(x) \quad \Leftrightarrow \quad A(x) + c = B(x) + c \quad (c \in \mathbf{R})$$
$$A(x) = B(x) \quad \Leftrightarrow \quad k \cdot A(x) = k \cdot B(x) \quad (k \in \mathbf{R} \ \wedge \ k \neq 0)$$

Sometimes, these rules are improperly extended and bring to the mistakes:

$$[A(x)]^2 = [B(x)]^2 \quad \Rightarrow \quad A(x) = B(x)$$

and, with reference to inequations:

$$A(x) < B(x) \quad \Leftrightarrow \quad k \cdot A(x) < k \cdot B(x) \quad (k \in \mathbf{R} \ \wedge \ k \neq 0)$$

We shall call this misconception *balance misconception* (it is not very different from the *misconception of linear mappings*, previously introduced: it can be considered an operational misconception). How does it happen that there are such mistakes in protocols by High School pupils (aged 14-19 years)?

Mentioned mistakes can be considered similar; there is a correct rule, and it is rather "simple": students accept it as a reliable one; so they seem to be induced to extend improperly this rule to cases different from the original one.

Concerning generalisations, let's now briefly point out some elements of the learning of function concept.

2. FUNCTIONS AND GENERALISATION

According to *property-oriented* approaches to function concept (Kieren, 1990), a function can be introduced and described with reference to its local properties (intersections, maximum and minimum points, vertical asymptotes etc.) and to its global properties (symmetry, periodicity, invertibility etc.).

Educational experience allows us to state that the study of local and global properties of a functions fundamental in order to characterise important classes

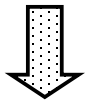
of functions: for instance, linear functions' Cartesian graphs are characterised by evident global properties and this allows us to identify a linear function as a function whose graph has got some considered global properties; and similarly as regard other important classes of functions, as quadratic functions, periodic functions etc.; on the contrary, main properties that characterise continuous functions (or derivable functions) are local.

Some experimental researches (Slavit, 1997, p. 272) pointed out that frequently pupils use either approaches based upon the consideration of a real correspondence (*action view*, *operational view*) or approaches based upon the consideration of some particular properties (*property-oriented*). For instance, when they study a function (we now make reference to Italian High School, 18-19 years old pupils), they sometimes consider either correspondences between some values of variable x and their $f(x)$, or global and local properties of the considered function. Form the semiotic point of view, the early approach is referred to numeric representations, the latter one to graphic representations, so to the visual register.

A *property-oriented* approach is useful in order to classify different functions. So previous considerations about *property-oriented* approaches can lead to processes based upon *particularization*: from the (general) concept of relation we consider the function concept, then a particular class of functions (for instance, linear functions). However, educational experience suggests an opposite path: a particular correspondence leads to its *generalization* in a whole class of functions in order to consider, finally, the general concept.

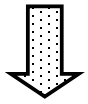
early consideration

of a concrete correspondence (e.g. a linear function)



first generalization

the class of linear functions (to whom the considered correspondence and other ones belong)



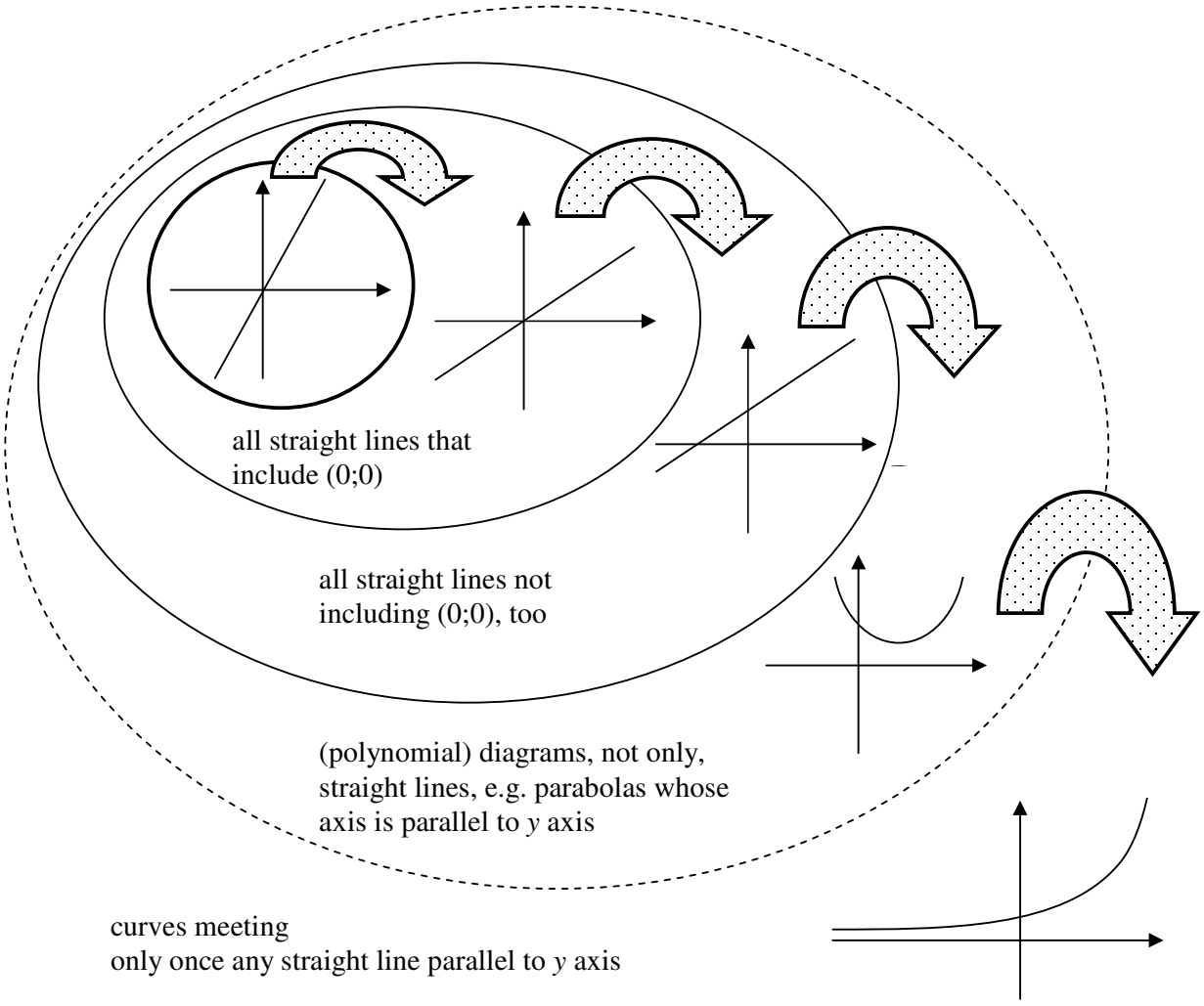
further generalization

"any" functions (e.g. linear and non-linear functions)

Every passage from a class (a set) of functions to another wider class brings us to consider new correspondences:

- having any *common* features with correspondences previously considered (for instance with reference to their Cartesian graphs);
- having any *different* features from correspondences previously considered (for instance with reference to their Cartesian graphs).

Let us underline once again the fundamental role of counterexamples: pupils must realize that the proposed generalization is a real one.



curves meeting
only once any straight line parallel to *y* axis

(this extension can be proposed in a graphic register or in a symbolic register)

Let's notice that it is not always easy to distinguish the classes by their graphic representations: for instance, we can hardly distinguish between

"polynomial functions" and "general functions" (in the previous figure the boundary line is... light). As regard graphics of polynomial functions, we can suggest the absence of asymptotes, but such property cannot be referred only to polynomial functions (e.g. let us consider the Cartesian graphic of $y = x+\cos x$: no asymptotes!). So sometimes a symbolic register can be more effective than a graphic one.

We have pointed out that the role of generalisation is quite important, for instance with reference to the introduction of function concept. But if we want to introduce a mathematical concept in a correct way, we must control the features of such generalisations.

In this paper, we shall analyse improper generalisations of "simple rules", with reference to High School students (students aged 16-19 years). We shall open our study with the investigation of some cases: a mistake about linear functions (the case of Sandra); a mistake about the solution of a quadratic equation (the case of Alberto); a mistake about Calculus (the case of Matteo). Then we shall present an experimental research about trigonometry.

3. SOME HIGH SCHOOL STUDENTS

3.1. Sandra and linear mappings

Sandra is an High School student of average mathematical skill, aged 17 years (4^{th} class of a *Liceo scientifico*, in Treviso, Italy); she wrote in a protocol the following solution of the equation (being $x \in \mathbf{R}$):

$$\log_e(x^2+7) = \log_e 7 \Rightarrow \log_e x^2+\log_e 7 = \log_e 7 \Rightarrow \log_e x^2 = 0 \Rightarrow x = 1$$

and (in the same protocol) the equality:

$$\sqrt{a^4+9\log_e e+\log_e 1} = \sqrt{a^4+9\cdot 1+0} = \sqrt{a^4+9} = a^2+3$$

So she considered the functions $x \to \log_e x$ and $x \to x^2$ as a linear mappings.

Sandra's interview took place in the classroom, in other pupils' presence:

Teacher: «First of all, let us consider the development of: $\log_e(x^2+7)$. Why did you write $\log_e (x^2+7) = \log_e x^2+\log_e 7$?»

Sandra: «It seemed to me a natural thing to do».

Teacher: «Why?».

Sandra: «I think that there is a property that states something like this».

Teacher: «There are no properties to simplify $\log_e(a+b)$. Do you remember any property about it?»

Sandra: «I don't remember, now: we studied the function $x \to \log_e x$ several months ago».

Teacher: «So why did you think that $\log_e(a+b) = \log_e a+\log_e b$?»

Sandra: «Well, it seems correct».

[The statement «it seems correct» is interesting: Sandra understood that her solution is incorrect, in teacher's opinion, but she noticed that *it seems* (at the moment of the interview) *correct*; she did not say *it seemed correct* (at the moment of the test): this suggests that, at the moment of the interview, she was still persuaded about the correctness of $\log_e(a+b) = \log_e a+\log_e b$].

Teacher: «And is it correct?»
Sandra (worried): «No».
Teacher: «Why? Did you change your mind?»
Sandra: «You told me I made a mistake. So I made something wrong».

[So Sandra hardly accepts the teacher's correction. Then the teacher proposes a counterexample].

Teacher: «Let us consider this example: $\log_e(e+1)$. We know that $\log_e e = 1$ and $\log_e 1 = 0$. Well, if now we should accept that $\log_e(a+b) = \log_e a+\log_e b$, we must write: $\log_e(e+1) = \log_e e+\log_e 1 = 1+0 = 1$. Do you think it is true?»
Sandra (after a moment): «No. It is false».
Teacher: «Why?»
Sandra: «Of course it is false: 1 is $\log_e e$ so it cannot be $\log_e(e+1)$».

[So Sandra implicitly states that the function $x \to \log_e x$ is injective. But we are not sure that this statement is based upon a deep knowledge of that function: did Sandra extend injectivity from a linear function, like $x \to kx$ ($k \neq 0$), to the function $x \to \log_e x$?]

Teacher: «So $\log_e(a+b)$ is not $\log_e a+\log_e b$. Do you agree?».
Sandra (calm): «Yes, of course. But *it seems* correct, doesn't it?»

[Then Sandra wrote the correct solution of the equation:

$$\log_e(x^2+7) = \log_e 7 \Rightarrow x^2+7 = 7 \Rightarrow x^2 = 0 \Rightarrow x = 0]$$

Teacher: «The solution is correct. Now let us see: $\sqrt{a^4+9}$. Why did you write $\sqrt{a^4+9} = a^2+3$?»
Sandra: «I thought: $\sqrt{a^4} = a^2$ and $\sqrt{9} = 3$».
Teacher: «Do you think your process is correct?»
Sandra: «Well, $\sqrt{a^4}$ is really a^2 and $\sqrt{9}$ is 3».
Teacher: «Of course: but is it correct to say that $\sqrt{a^4+9} = a^2+3$?»
Sandra (smiling): «I guess the answer is no, isn't it? But how can I simplify $\sqrt{a^4+9}$?»
Teacher: «There are no properties to simplify $\sqrt{a^4+9}$»

Sandra: «No properties about $\log_e(a+b)$, no properties about $\sqrt{a^4+9}$. But it was possible to solve the equation: and now, how can I simplify $\sqrt{a^4+9}$?»

Teacher: «You cannot simplify it».

Sandra: «So when I find $\sqrt{a^4+9}$ I must stop».

Teacher: «Of course: you cannot continue: $\sqrt{a^4+9}$ is the final result of your exercise».

Sandra (worried): «Agreed».

Sandra does not seem quite persuaded: she explicitly noticed that the equation $\log_e(x^2+7) = \log_e 7$ can be solved without writing $\log_e(x^2+7) = \log_e x^2 + \log_e 7$; so, in this case, *there is* a correct route to be taken, instead of the wrong route. The case $\sqrt{a^4+9}$ seems a different one; Sandra asked *twice*: «how can I simplify $\sqrt{a^4+9}$?» It is impossible to «continue» this exercise without writing $\sqrt{a^4+9} = a^2+3...$

It is very interesting to consider the final part of the interview:

Teacher: «For example, do you think that $e^{a+b} = e^a+e^b$?»

Sandra (lit up with joy!): «Oh no, no! I know very well that $e^{a+b} = e^a \cdot e^b$!»

Sandra will not fall in mistakes like $e^{a\pm b} = e^a \pm e^b$: in these cases *she knows* some simple rules to simplify $e^{a\pm b}$ ($e^{a+b} = e^a \cdot e^b$ and $e^{a-b} = e^a / e^b$), so she is not forced to extend (improperly) other rules. However, let us underline that this situation is not referred to an exercise, so it is not influenced by the didactic contract: the last answer can be influenced by the experimental contract (see: Schubauer Leoni, 1988; Schubauer Leoni & Ntamakiliro, 1994).

3.2. Alberto and equations

Alberto is an High School student of average mathematical skill, aged 16 years (3rd class of *Liceo scientifico*, in Treviso, Italy); he wrote in a protocol the following solution (being $x \in \mathbf{R}$):

$$5x^2 = 20 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

Of course, the teacher underlined that this solution is wrong because it "forgets" the root $x = -2$. He showed the following (correct) solution:

$$5x^2 = 20 \Rightarrow x^2 = 4 \Rightarrow x^2 - 4 = 0 \Rightarrow$$
$$\Rightarrow (x+2)(x-2) = 0 \Rightarrow x = -2 \vee x = 2$$

Alberto's interview took place in the classroom, in other pupils' presence:

Teacher: «What about this solution?»

Alberto: «Of course it is correct, but it is rather strange, difficult for me. I did not think that the rule $a^2–b^2 = (a+b)(a–b)$ is necessary to solve an equation».

Teacher: «It is not absolutely necessary: it is enough to remember that $2^2 = (–2)^2 = 4$. Please, describe entirely your solution».

Alberto: «First of all, I divided both members by 5; then I calculated the square roots of both members: this is easy. And it is very easy to be kept in mind...»

So Alberto states that he chose the solution «easy to be kept in mind»; he «did» the same «operations» in both members of the equation, hoping to obtain a new equation equivalent to the original one: the mistake seems clear, now.

Well, is everything clear, after the mentioned interview?

In another protocol, *just a month later*, Alberto wrote:

$$y^2–x^2 = 2x+1 \quad \Rightarrow \quad y^2 = x^2+2x+1 \quad \Rightarrow \quad \sqrt{y^2} = \sqrt{(x+1)^2} \quad \Rightarrow \quad y = x+1$$

So the *balance misconception* is very strong, lasting: even the teacher's correction was not effective enough to overcome it, in Alberto's mind.

3.3. Another example

We underline once again that the mistakes previously considered can be found in several school levels. Let us briefly see the case of Matteo, an High School student aged 18 years (5^{th} class of *Liceo scientifico*, in Treviso, Italy): in a protocol, he wrote down the following correct process:

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{(x+1)}{(x-2)} = -2$$

that can be represented as:

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f(x)\big/h(x)}{g(x)\big/h(x)}$$

But in the same protocol, just few rows later, Matteo... extended it to the following wrong process:

$$\lim_{x\to+\infty} \frac{\sqrt{x}}{x+1} = \lim_{x\to+\infty} \frac{\left(\sqrt{x}\right)^2}{(x+1)^2} = \lim_{x\to+\infty} \frac{x}{x^2+2x+1} = 0$$

(where... only the final result is correct!) that can be represented as:

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{\left[f(x)\right]^2}{\left[g(x)\right]^2}$$

Can we consider it only a "casual" mistake? In our opinion, in this situation we can underline the clear influence of the (correct) habit to simplify some fractions in order to calculate a limit; the student tried to extend (incorrectly) this process, so we can point out the presence of a misconception similar to the *balance misconception* (applied to fractions).

We shall not examine closely Matteo's mistake: we wanted just underline that some mistakes that can be connected to the *balance misconception* are present in students aged 18-19 years (as regards the learning of the notion of limit, and in particular some important misconceptions, see for example: Cornu, 1980; Davis & Vinner, 1986; Dimarakis & Gagatsis, 1996).

4. FROM ARITHMETICS TO ALGEBRA: EDUCATIONAL ROOTS OF ALGEBRAIC THOUGHT

Previously, we pointed out that students sometimes improperly extend «simple» rules. Sandra (paragraph 3.1) seemed to look for any rule to simplify some expressions, so she considered the functions $x\to\log_e x$ and $x\to x^2$ as linear mappings; but let us remember the final part of her interview, too: it seems that mistakes like $e^{a\pm b}=e^a\pm e^b$ are not frequent, because in these cases *students know* some rules to simplify $e^{a\pm b}$ ($e^{a+b}=e^a.e^b$ and $e^{a-b}=e^a/e^b$): so they are not forced to extend improperly other rules (see the following paragraph).

From a technical point of view, we should say that the examined mistakes are based upon algebraic weakness (let us consider for example Alberto's mistake and the *balance misconception*, paragraph 3.2, or Matteo's mistake, paragraph 3.3; let us quote once again: Tietze, 1988; Malle, 1993). It is interesting to examine briefly some educational roots of algebraic thought.

F. Arzarello, L. Bazzini and G. Chiappini notice:

> «Several Authors state that the roots of algebraic thought can be pointed out in the effort to consider a computational process in a quite general way» (Arzarello, Bazzini & Chiappini, 1994, p. 10).

So Algebra itself is often introduced as a generalisation: for example, we generalise arithmetical operations by some algebraic process; and this «replacement» of Arithmetics by Algebra is sometimes a source of obstacles. Y. Chevallard writes:

> «For several generations, Arithmetics was the green Paradise of [...] the spirit opening to an marvellous intellectual activity [...] So an only too well learnt Arithmetics became an intellectual, affective and ideological obstacle

against its overcoming» (Chevallard, 1989, p. 15; see moreover: Vergnaud, Cortes & Favre-Ortigue, 1997, p. 253).

The role of formal transformations is important: L. Bazzini notices:

«Students' answers to questions about equivalence of equations (or inequalities) are highly influenced by presence or absence of formal transformations. This [...] entails a careful reflection about cognitive processes in learning of Algebra» (Bazzini, 1995, p. 44; see moreover: Linchevski & Sfard, 1991; Sfard & Linchevski, 1992; Arcavi, 1994; Cortés, 1994).

Students are enabled to interpret algebraic symbols and processes *not* only from a syntactic point of view (Burton, 1988; Tall, 1990).

So we can resume: in the examined students' mistakes, obstacles are related to algebraic weakness: so some pupils do not aware several basic algebraic techniques; moreover, we must underline the presence of obstacles related to affective sphere: students know a "simple" rule, they use successfully in many cases, so they associate those rules to good performances; when they have no rules to use in some problems' resolutions, they improperly extend those rules, hoping to have, once again, good performances. By the following tests we wanted to point out that affective aspect is fundamental to settle this situation.

5. A BRIEF EXPERIMENTAL RESEARCH

5.1. Method of tests

Two tests were proposed to students belonging to two 4^{th} classes and two 5^{th} classes of a *Liceo scientifico* (High School; pupils aged 17-19 years) in Treviso, Italy, total 95 students (two 4^{th} classes: 23 and 24 pupils respectively; two 5^{th} classes: 26 and 22 pupils respectively); we shall identify them by group A (the first 4^{th} and 5^{th} classes, total 49 pupils) and group B (the second 4^{th} and 5^{th} classes, total 46 pupils; all students had the same mathematics teacher; their curricula were standard: they knew basic elements of trigonometry; in particular, they knew the equality: $\sin^2 x+\cos^2 x = 1$).

The first test (A) was proposed to the 49 pupils of the group A (4^{th} class: 23 pupils; 5^{th} class: 26 pupils):

A) You know that $\sin^2 x+\cos^2 x = 1$ is true for every x. Is the equality:

$\sin^4 x+\cos^4 x = 1$

true for every x?

Time: 1 minute (we wanted that students examine the problem "at a glance"). No textbooks or electronic calculators allowed.

The second test (B) was proposed to the 46 pupils of the group B (4^{th} class: 24 pupils; 5^{th} class: 22 pupils):

> B) Let $a^2+b^2 = 1$. Is the equality:
>
> $a^4+b^4 = 1$
>
> true for every a and for every b such that $a^2+b^2 = 1$?

Time: 1 minute. No textbooks or electronic calculators allowed.

By these tests we wanted to examine the influence of the well-known rule $\sin^2x+\cos^2x = 1$ in the interpretation of the (incorrect) equality $\sin^4x+\cos^4x = 1$ (test A). The test B is based upon the incorrect statement $a^2+b^2 = 1 \Rightarrow a^4+b^4 = 1$, that can be considered technically equivalent to the problem expressed in the test A, but it has no reference with the basic rule $\sin^2x+\cos^2x = 1$.

5.2. Results of the tests and considerations

Group A						
	4^{th} class (23 students)		5^{th} class (26 students)		Total (49 students)	
Yes (true)	9	39%	12	46%	21	43%
No (false)	10	44%	8	31%	18	37%
No answer	4	17%	6	23%	10	20%

Group B						
	4^{th} class (24 students)		5^{th} class (22 students)		Total (46 students)	
Yes (true)	4	17%	2	9%	6	13%
No (false)	19	79%	16	73%	35	76%
No answer	1	4%	4	18%	5	11%

These results show a clear difference: as regards the group A, 43% of the students stated that $\sin^4x+\cos^4x = 1$; probably, they were influenced by the (correct) rule $\sin^2x+\cos^2x = 1$. As regards the group B, in fact, only 13% of the students stated that $a^2+b^2 = 1 \Rightarrow a^4+b^4 = 1$.

Let us remember that several students (19 out of 35 students that answered *no* or *false* to the question of the test B) noticed that $a^2+b^2 = 1 \Rightarrow a^4+b^4 = 1$ can be true only for some particular values of a and b.

5.3. Justifications given by students

Several students gave interesting justifications; as regards the students that answered *yes* or *true* to the question of the test A, let us remember:

«I saw $\sin^4 x+\cos^4 x = 1$ and I immediately thought that $\sin^4 x = (\sin^2 x)^2$ and $\cos^4 x = (\sin^2 x)^2$, so I concluded that $\sin^4 x+\cos^4 x = 1$ is true» (Aldo, 4th class); 9 justifications (test A) and 4 justification (test B) are similar to this one.

In Aldo's justification we can clearly point out the presence of the *misconception of linear mappings*.

«Of course, I realise I've made a big mistake, I do not know the reason: I remembered the famous trigonometric rule $\sin^2 x+\cos^2 x = 1$ and I thought that it could be true for 4, too. But why? I did not remember any other rules about $\sin^4 x+\cos^4 x = 1$ so I tried to apply the one and only rule I could remember» (Anna, 5th class).

Anna's justification is not very different from Sandra's one (see paragraph 3.1): she had no rules for $\sin^4 x+\cos^4 x = 1$, so she extended a well-known rule...

As regards students that answered *no* or *false* to the question of the test B, let us remember:

«The equality is false: a^4 and b^4 are not equal to a^2 and b^2: it is true only for some particular cases» (Antonio, 5th class).

As previously remembered, 19 out of 35 students that answered *no* or *false* to the question of the test B underlined that $a^2+b^2 = 1 \Rightarrow a^4+b^4 = 1$ can be true only for some particular values of a and b.

We can conclude that the presence itself of a "rule" just similar to the "famous" formula $\sin^2 x+\cos^2 x = 1$ induced many students to refer to it: in the test A the well-known rule is explicitly present; on the contrary, in B the incorrect statement $a^2+b^2 = 1 \Rightarrow a^4+b^4 = 1$ is not referred to any "famous" rule.

6. GENERAL CONCLUSIONS

6.1. Roots of some misconceptions

Results of the experimental research previously presented needs some remarks.

First of all, let us remember that several researches showed that the different representations of a problem are very important as regards students' behaviour in problem solving (see: Fischbein, Tirosh & Hess, 1979; Silver, 1986; Arcavi, Tirosh & Nachmias, 1989; see moreover: Gagatsis & Thomaidis, 1995).

The situations previously described show that a "simple" rule is often seen as a natural and a reassuring one. So, from an affective point of view, too, some

students are induced to apply it to a lot of cases, without particular controls: of course, this can cause dangerous mistakes.

Many students try to extend a well-known rule when they do not know specific rules to solve a problem. Let us remember, for example, that it is well known that pupils are afraid of problems "without a result" (Baruk, 1985; Micol, 1991; Schubauer Leoni & Ntamakiliro, 1994; a systematic classification of "impossible" problems can be found in: D'Amore & Sandri, 1993). So we can state that this fear brings many students to solve the considered problem by a familiar, reliable rule: unfortunately sometimes this rule cannot be applied to the considered case.

6.2. How can we overcome these misconceptions?

As we shall see, it is not easy to overcome these misconceptions. We should say that the role of counterexamples is important to make students aware of incorrect answers and of their conflicting ideas.

As regards the mistake $\sin^4 x + \cos^4 x = 1$ (for every $x \in \mathbf{R}$, paragraph 4.1, test A), it is easy to show directly that if we consider the case $x = \frac{\pi}{4}$, we have:

$$\sin^4\left(\frac{\pi}{4}\right) + \cos^4\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^4 + \left(\frac{\sqrt{2}}{2}\right)^4 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 1$$

Moreover, if we consider the *misconception of linear mappings*, we can remember the Theorem of Pythagoras (a, b, c are measures of three sides of a triangle having a right angle, a is hypotenuse's measure), by which we can write:

$$a = \sqrt{b^2 + c^2}$$

Of course, if we write:

$$a = \sqrt{b^2 + c^2} = \sqrt{b^2} + \sqrt{c^2} = b + c$$

we should state that the sum of two sides of a triangle is equal to the third one, and this is clearly an absurd statement.

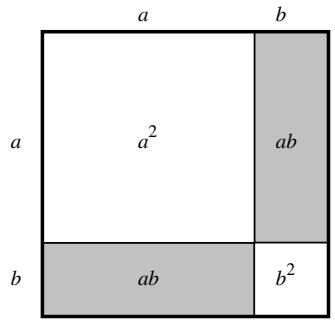
As regards counterexamples, visualisation can be very important. Some Authors, in the last years, worked about matters connected to visualisation. R. Duval notices that «mathematical objects are not directly accessible to the perception [...] as objects generally said 'real' or 'physical'»; so he states that «different semiotic representations of a mathematical object are absolutely necessary» (Duval, 1993, p. 37). The important presence of different registers of representation is, in Duval's opinion, remarkable:

«The cognitive functioning of human thought is inseparable from the existence of a variety of semiotic registers of representation. If we call *sémiosis* the learning of the production of a semiotic representation and *noésis* the conceptual learning of an object, we must affirm that *sémiosis* is inseparable from *noésis*» (Duval, 1993, pp. 39-40).

A well-known work by E. Fischbein is devoted to visual representation of mathematical objects and to its great importance in mathematics education; Fischbein states that

«the integration of conceptual and figural properties in unitary mental structures, with the predominance of the conceptual constraints over the figural ones, is not a natural process. It should constitute a continuous, systematic and main preoccupation of the teacher» (Fischbein, 1993, p. 156; as regards functions, see moreover: Vinner, 1983, 1987 and 1992).

The following figure is referred to the 4^{th} Proposition of Euclidean Geometric Algebra (Euclid, 1970, p. 163; from historical point of view, see for example: Boyer, 1968; Kline, 1972; van der Waerden, 1983; Anglin, 1994).



Nowadays the 4^{th} Proposition of Euclidean Geometric Algebra can be expressed by:

$$(a+b)^2 = a^2+b^2+2ab$$

but in *Elements* only the previous picture gives the proof of this proposition. The mistake that identifies improperly $(a+b)^2$ in a^2+b^2 (without the so-called "double product", according to the *misconception of linear mappings*) is nearly impossible if the visual representation is correctly considered (Kaldrimidou, 1987; Bagni, 1997).

Nevertheless, we cannot say that the use of counterexamples is always conclusive: *the effect of counterexamples with students is often weak since they are not able to interpret given counterexamples in an adequate way*.

As regards this important point, let us remember once again Alberto's case; the pupil wrote: $x^2 = 4 \Rightarrow x = 2$. In order to correct this mistake, the teacher underlined that: $2^2 = (-2)^2 = 4$; but clearly this correction was not effective enough to overcome completely the *balance misconception*, in Alberto's mind. In fact the pupil simply said: «I calculated the square roots of both members». So Alberto understood teacher's statement, but he was not able to interpret the correction in the sense of a real counterexample, strictly related to his previous mistake: he was not able to connect effectively $2^2 = (-2)^2 = 4$ (square powers) to the correct solution, $x^2 = 4 \Rightarrow x = -2 \vee x = 2$ (square roots).

Then it seems that some misconceptions are really lasting: although they can be sources of inconsistencies in student's minds, they reoccur and their effects can be pointed out several times (as regards the presence of conflicting answers and of ideas that are incompatible with each other, see for example: Tall, 1990; Tsamir & Tirosh, 1992; let us remember that several researches showed that sometimes students do not realise the presence of conflicting answers: Stavy & Berkovitz, 1980; Hart, 1981: for example, the persistence of different sorts of algebra errors in pupils aged 11-18 years is proved: Matz, 1982; and sometimes the presence of ideas that are incompatible with each other is not considered completely illicit: Schoenfeld, 1985; Tirosh, 1990).

6.3. Final reflections

We do not think that the obstacles previously examined can be considered as epistemological ones or (only) as educational ones (see the classification in: Brousseau, 1983; Vergnaud, 1989, pp. 168-169). If we consider them as educational obstacles, we must underline that the influence of affective aspect is surely remarkable. Then, in our opinion, they can be regarded as *affective obstacles*, too: so it is difficult to overcome them completely just by educational means, like showing of counterexamples (D'Amore & Martini, 1997).

Of course, we must underline that analogical reasoning should not be too quickly dismissed: in fact, many mathematicians used and use it as one of the main ways for creating new mathematics! However, the really different propensity for self-correction should be considered, when we compare research mathematicians and young students: for example, frequently mathematicians employ analogical reasoning in formulation of a conjecture whose logical soundness must be verified; on the other hand, generally students do not perform this meta-discursive monitoring.

REFERENCES

Anglin, W.S.: 1994, *Mathematics. A Concise History and Philosophy*, Springer, Berlin.

Arcavi, A.; Tirosh, D., & Nachmias, R.: 1989, The effects of exploring a new representation on prospective teachers' conception of functions, Vinner, S. (Ed.) *Science and mathematics teaching: Interaction between research and practice*, Hebrew University, Jerusalem, Israel.

Arcavi, A.: 1994, Symbol sense: informal sense-making in formal mathematics, *For the learning of mathematics*, 14, 3, 24-35.

Arzarello, F.; Bazzini, L. & Chiappini, G.: 1994, *L'Algebra come strumento di pensiero. Analisi teorica e considerazioni didattiche*, Progetto strategico del CNR: Tecnologie e innovazioni didattiche, Quaderno n. 6.

Bagni, G.T.: 1997, La visualizzazione nella scuola secondaria superiore, *L'insegnamento della matematica e delle scienze integrate*, 20B, 4, 309-335.

Bagni, G.T.: 2000, «Simple» rules and general rules in some High School students' mistakes, *Journal fur Mathematik Didaktik*, 21 (2000), 2, 124-138.

Baruk, S.: 1985: *L'âge du capitain*, Seuil, Paris.

Bazzini, L.: 1995, Equazioni e disequazioni: riflessioni sul concetto di equivalenza, Bazzini, L. (Ed.), *La didattica dell'Algebra nella scuola secondaria superiore*, Atti V Convegno Internuclei per la scuola superiore, Pavia, 16-18 marzo 1995, 44-53.

Boyer, C.B.: 1968, *A History of Mathematics*, John Wiley and Sons, Inc., New York.

Brousseau, G.: 1983, Ostacles epistemologiques en mathématiques: *Recherches en Didactique des Mathématiques*, 4, 2.

Brousseau, G.: 1986, Fondaments et méthods de la didactique des mathématiques, *Recherches en Didactique des Mathématiques*, 7, 2, 33-115.

Burton, M.B.: 1988, A linguistic basis for students' difficulties with Algebra, *For the learning of mathematics*, 8.

Chevallard, Y.: 1989, *Arithmetique, algebre, modelisation*, IREM d'Aix-Marseille.

Cornu, B.: 1980, Interference des modeles spontanes dans l'apprentissage de la notion de limite, *Cahier du Seminaire de Didactique des Mathematiques et de l'Informatique*, 8, 57-83.

Cortés, A.: 1994, Modélisation cognitiviste: invariants opératoires dans la résolution des équations, Artigue, M.; Gras, R.; Laborde, C. & Tavignot, P. (Eds.), *Vingt ans de didactique des mathématiques en France*, La pensée sauvage, Paris, 210-217.

D'Amore, B. & Sandri, P.: 1993, Una classificazione dei problemi cosiddetti impossibili: *La matematica e la sua didattica*, 3, 344-347.

D'Amore, B. & Martini, B.: 1997, Contratto didattico, modelli mentali e modelli intuitivi nella risoluzione di problemi scolastici standard, *La matematica e la sua didattica*, 2, 150-175.

D'Amore, B. & Sandri, P.: 1998, Le risposte degli allievi a problemi di tipo scolastico standard con un dato mancante: *La matematica e la sua didattica*, preprint.

Davis, P. & Vinner, S.: 1986, The Notion of Limit: Some Seemingly Unavoidable Misconception Stages: *Journal of Mathematical Behavior*, 5, 281-303.

Dimarakis, I. & Gagatsis, A.: 1996, The limit concept; Difficulties-obstacles of Students' Undestanding, Gagatsis, A. & Rogers, L. (Eds.), *Didactics and History of Mathematics*, Erasmus, Thessaloniki.

Duval, R.: 1983, L'ostacle du dedoublement des objects mathématiques, *Educational Studies in Mathematics*, 14, 385-414.

Duval, R.: 1993, Registres de répresentation sémiotique et fonctionnement cognitif de la pensée, *Annales de Didactique et de Sciences Cognitives*, 5, IREM, Strasbourg.

Euclid: 1970, *Elementi*, Frajese, A. & Maccioni, L. (Eds.), UTET, Torino.

Fischbein, E.; Tirosh, D. & Hess, P.: 1979, The intuition of infinity: *Educational Studies in Mathematics*, 10, 3-10.

Fischbein, E.: 1993, The theory of figural concepts, *Educational Studies in Mathematics*, 24, 139-162.

Gagatsis, A. & Thomaidis, J.: 1995, Eine Studie zur historischen Entwicklung und didactischen Transposition des Begriffs "absoluter Betrag", *Journal für Mathematik-Didactik*, 16, 1/2, 3-46.

Hart, K. (Ed.): 1981, *Children's understanding of mathematics*, Murray, London.

Kaldrimidou, M.: 1987, *Images mentales et représentations en mathématiques chez les mathématiciens et les étudiants en mathématiques*, Thèse 3éme cycle, Université Paris 7, Paris.

Kieren, T.E. (1990), Understanding for teaching for understanding, *The Alberta Journal of Educational Research*, 36 (3), 191-201.

Kline, M.: 1972, *Mathematical thought from ancient to modern times*, Oxford University Press, New York.

Linchevski, L. & Sfard, A.: 1991, Rules without reasons as processes without objects, the case of equations and inequalities, Furinghetti, F. (Ed.), *Proceedings of PME XV*, Assisi, 2, 317-324.

Malle, G.: 1993, *Didaktische Probleme der elementaren Algebra*, Vieweg, Braunschweig.

Markovitz, Z.; Eylon, B. & Bruckheimer, N.: 1986, Functions today and yesterday, *For the learning of mathematics*, 6 (2), 18-24.

Matz, M.: 1982, A process model for high school algebra errors, Sleeman & Brown (Eds.), *Intelligent tutoring systems*, Academic, London.

Micol, G.: 1991, I problemi "impossibili", *La matematica e la sua didattica*, 4, 45-49.

Schoenfeld, A.: 1985, *Mathematical problem solving*, Academic, New York.

Schubauer Leoni, M.L.: 1988, L'interaction expérimentateur-sujet à propos d'un savoir mathématique: la situation de test revisitée, Perret-Clermont, A.N., & Nicolet, M. (Eds.), *Interagir et connaître*, DelVal, Cousset, Suisse.

Schubauer Leoni, M.L. & Ntamakiliro, L.: 1994, La construction de réponses à des problèmes impossibles, *Revue des sciences de l'éducation*, XX, I, 87-113.

Silver, E.: 1986, Using conceptual and procedural knowledge: A focus on relationships, J. Hilbert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*, 181-198, Lawrence Erlbaum, Hillsdale (NJ).

Sfard, A.: 1991, On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coins, *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. & Linchevski, L.: 1992, Equations and inequalities. Processes without objects?, *Proceedings of PME XVI*, Durham, 3, 136.

Slavit, D. (1997), An alternate route to reification of function, *Educational Studies in Mathematics* 33, 259-281.

Stavy, R. & Berkovitz, B.: 1980, Cognitive conflict as a basis for teaching qualitative aspects of the concept of temperature, *Science Education*, 28, 305-313.

Tall, D.: 1990, Inconsistencies in the learning of Calculus and Analysis, *Focus on Learning Problems in Mahematics*, 12, 49-64.

Tall, D.: 1992, Mathematical processes and symbols in the mind, Karian, Z.A. (Ed.), *Symbolic computation in undergraduate mathematics education*, MAA Notes 24, 57-68

Tietze, U.: 1988, Schülerfehler und Lernschwierigkeiten in Algebra und Arithmetik, *Journal für Mathematik-Didaktik* 9, 2/3, 163-204.

Tirosh, D.: 1990, Inconsistencies in students' mathematical constructs, *Focus on Learning Problems in Mathematics*, 12, 111-129.

Tsamir, P. & Tirosh, D.: 1992, Students' awareness of inconsistent ideas about actual infinity, *PME XVI*, 90-97, Durham (NH).

Vergnaud, G.: 1989, Difficultés conceptuelles, erreurs didactiques et vrais obstacles épistémologiques dans l'apprentissage des mathématiques: Bednarz, N. & Garnier, C., *Construction des savoirs*, Cirade, Ottawa, 33-40.

Vergnaud, G.; Cortes, A. & Favre-Ortigue, P.: 1997, Introduzione dell'algebra ai principianti "deboli": problemi epistemologici e didattici, *La matematica e la sua didattica*, 3, 253-271

(*Actes du Colloque de Sèvres: Didactique et acquisition des connaissances scientifiques*, 1987, 259-279).

Vinner, S.: 1983, Concept definition, concept image and the notion of function: *International Journal for Mathematical Education in Science and Technology*, 14, 3, 293-305.

Vinner, S.: 1987, Continuous functions-images and reasoning in College students: *Proceedings PME 11*, II, Montreal, 177-183.

Vinner, S.: 1992, Function concept as prototype for problems in mathematics, Harel, G. & Dubinsky, E. (Eds.), *The concept of Function: aspects of Epistemology and Pedagogy*, MAA Notes, 25, 195-213.

Van der Waerden, B.L.: 1983, *Geometry and Algebra in Ancient Civilizations*, Springer, Berlin.