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Ancient *Zara* game and teaching of Probability: an experimental research in Italian High School

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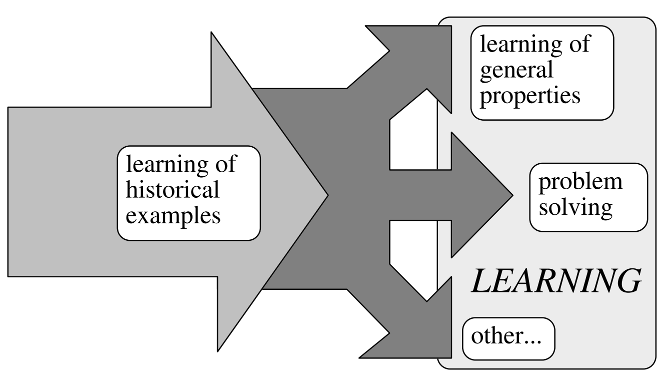
Abstract. This work is the beginning of a research activity on a first approach to probability from an informal point of view. We presented to students aged 17-18 a class activity based upon a simplified version of an ancient game named *Zara*. This game is famous in Italy because it is quoted by Dante Alighieri in *Purgatorio*, VI, 1-3, and by Galileo Galilei. It is interesting to underline that students adopt Laplace first and second principles without knowing them; according with Laplace ideas, several students made mistakes in non-symmetric probabilities. In this research the use of historical examples, combined with activity linked to everyday life, allows an effective approach to concepts of Probability.

INTRODUCTION

This work can be considered as the beginning of a research activity on a first approach to Probability (from an informal point of view), based upon an historical example. We shall consider some experimental results (some of such results were given and analyzed in: Bagni & Cecchini, 1999) in order to evaluate the effectiveness of some uses of History of Mathematics into Mathematics Education in the field of Probability.

We considered the following model:



(quoted in: Bagni & Cecchini, 1999, too).

In fact, by the consideration of historical examples, we operate on teaching to improve its quality; but some reactions, in pupils' minds, are just plausible, they are not quite sure. We proposed stimulation in the historical sphere, so in this sphere students will "learn": however the knowledge so achieved would *not* be bordered in the historical sphere.

So evolution to different spheres is quite necessary: the cognitive *transfer* (see for instance: Feldman & Toulmin, 1976); the problem that can limit the efficacy of Mathematics Education based upon historical examples is the following: *if we operate (only) on teaching, are we sure that the cognitive transfer will take place?* (D'Amore & Frabboni, 1996, pp. 97-98).

First of all, let us remember that, from the historical point of view, too, the introduction of fundamental concepts of Probability was slow and hard (Daston, 1980; Lakoma, 1998a and 1998b; as regards original texts see for example: Smith, 1959, pp. 546-604): historically, the introduction of the notion of Probability took place with reference to studies about games (Todhunter, 1965; Maistrov, 1974; as regards recent history, see: Kolmogorov & Yushkevich, 1992); and let us underline that sometimes J. R. d'Alembert (1717-1783) himself was not able to recognise symmetric (or non-symmetric) cases.

In this work we shall not give a full presentation of researches upon the didactic introduction of Probability (see for example the interesting summary in: Gagatsis, Anastasiadou & Bora-Senta, 1998). According to E. Fischbein (1975 and 1984; Fischbein, Nello & Marino, 1991) teaching of Probability would begin with reference to pupils aged 12-14 years; but sometimes, for instance in Italian School, pupils' approach to main concepts of Probability is rather late, so it is really meaningful to investigate the didactic approach to these concepts related to older pupils, for instance aged 17-18 years.

So we shall consider some High School pupils that did not receive a formal presentation of Probability, in order to point out their reactions and to underline obstacles they have to deal with. In particular, we shall investigate the effectiveness of a well-known historical example.

AN HISTORICAL EXAMPLE: *ZARA* GAME AND THE DOUBTS OF *GENTILUOMINI FIORENTINI*

Quando si parte il giuoco della zara
Colui che perde si riman dolente
Repetendo le volte, e tristo impara

Dante Alighieri, Purgatorio, VI, 1-3

Dante Alighieri (1265-1321) quoted in his *Purgatorio* an ancient popular Italian game: in the *Zara* game three dice are cast and the player tried to guess the total score so obtained (from 3 to 18). Of course, the results:

$$3 = 1+1+1 \qquad \text{and} \qquad 18 = 6+6+6$$

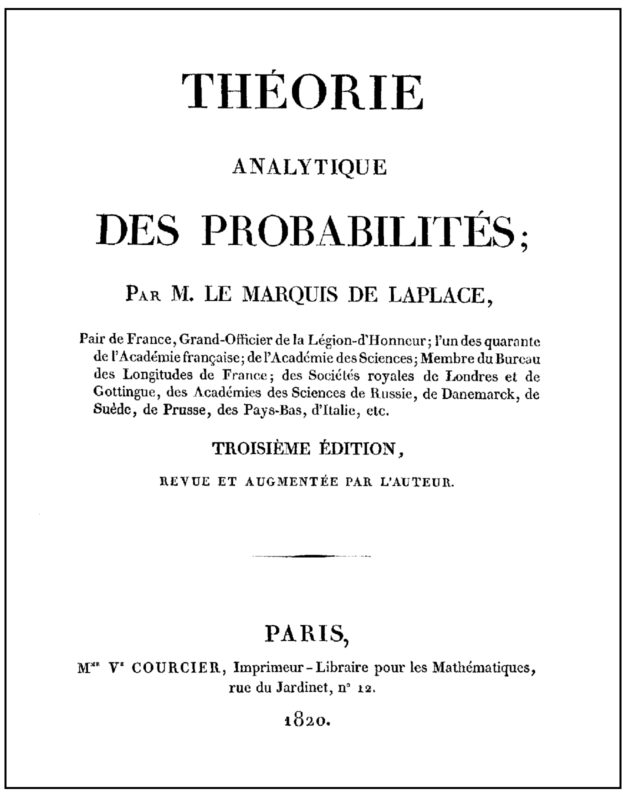
are not probable scores, because they can be obtained only by one combination; on the contrary, other scores, like 9, 10, 11, 12, ... are probable because they can be obtained by several different combinations (for instance: 9 = 1+2+6 = 1+3+5 = 1+4+4 = 2+2+5 etc.; of course we cannot say that Dante Alighieri was a… mathematician interested in Probability! See: D'Amore, 1994, p. 53).

More than three centuries later, Galileo Galilei (1564-1642) pointed out some interesting remarks about the *Zara* game (*Opere*, XIV; the original Italian text is quoted in: Bottazzini, Freguglia & Toti Rigatelli, 1992, pp. 344-347): he answered to some *Gentiluomini Fiorentini* (Florence gentlemen) that noticed that, in *Zara* game, scores 10 and 11 seem more frequent than scores 9 and 12. This remark can be considered… strange, if we keep in mind that:

9	=	1+2+6	=	1+3+5	=	1+4+4	=	
	=	2+2+5	=	2+3+4	=	3+3+3		
10	=	1+3+6	=	1+4+5	=	2+2+6	=	
	=	2+3+5	=	2+4+4	=	3+3+4		
11	=	1+4+6	=	1+5+5	=	2+3+6	=	
	=	2+4+5	=	3+3+5	=	3+4+4		
12	=	1+5+6	=	2+4+6	=	2+5+5	=	
	=	3+3+6	=	3+4+5	=	4+4+4		

LAPLACE'S *GENERAL PRINCIPLES*

Galileo's answer to *Gentiluomini Fiorentini* can be usefully compared with *General Principles of Probability* stated by Pierre Simon de Laplace (1749-1827) in his famous *Essay philosophique sur les probabilités* (1814); let us quote Laplace's own words:



THÉORIE

ANALYTIQUE

DES PROBABILITÉS;

PAR M. LE MARQUIS DE LAPLACE,

Pair de France, Grand-Officier de la Légion-d'Honneur; l'un des quarante de l'Académie française; de l'Académie des Sciences; Membre du Bureau des Longitudes de France; des Sociétés royales de Londres et de Gottingue, des Académies des Sciences de Russie, de Danemarck, de Suède, de Prusse, des Pays-Bas, d'Italie, etc.

TROISIÈME ÉDITION,

REVUE ET AUGMENTÉE PAR L'AUTEUR.

PARIS,

Mme Ve COURCIER, Imprimeur-Libraire pour les Mathématiques, rue du Jardinet, n° 12.

1820.

Third edition (Paris, 1820) of Laplace's *Theorie Analytique des Probabilités*

"1st Principle. It is the definition of Probability itself, that [...] is the ratio of the number of favourable cases and the number of all possible cases. 2nd Principle. It needs that all different cases are equally possible. If they are not so, it needs to find the respective possibilities, and this is one of the most difficult points of all the Theory" (Laplace, 1820).

Galileo seems to… know this text (written 170 years after Galileo's death!) and he simply underlines that probabilities of the scores remembered by *Gentiluomini Fiorentini* are *not* symmetric.

In today's language, if $\left(\frac{1}{6}\right)^3 = \frac{1}{216}$ is the probability of a single combination obtained when we cast our three dice, we can say that:

- the probability of a score given by the sum of three equal numbers is: $\frac{1}{216}$;
- the probability of a score given by the sum of two equal numbers and a different one is: $3 \cdot \frac{1}{216} = \frac{3}{216}$;
- the probability of a score given by the sum of three different numbers is $6 \cdot \frac{1}{216} = \frac{6}{216}$.

So probabilities of scores remembered by *Gentiluomini Fiorentini* are:

9	$\frac{6}{216}+\frac{6}{216}+\frac{3}{216}+\frac{3}{216}+\frac{6}{216}+\frac{1}{216} = \frac{25}{216}$
10	$\frac{6}{216}+\frac{6}{216}+\frac{3}{216}+\frac{6}{216}+\frac{3}{216}+\frac{3}{216} = \frac{27}{216}$
11	$\frac{6}{216}+\frac{3}{216}+\frac{6}{216}+\frac{6}{216}+\frac{3}{216}+\frac{3}{216} = \frac{27}{216}$
12	$\frac{6}{216}+\frac{6}{216}+\frac{3}{216}+\frac{3}{216}+\frac{6}{216}+\frac{1}{216} = \frac{25}{216}$

Of course, we can clearly conclude that scores 10 and 11 are more probable than scores 9 and 12.

AN EXPERIMENRAL RESEARCH

We considered an experimental research (a part of results are given in: Bagni & Cecchini, 1999) and we examined some High School pupils, belonging to four classes (4th *Liceo scientifico*, 86 pupils aged 17-18 years), in Treviso, Italy. As we previously noticed, they did not know a formal introduction to Probability.

We considered a simplified version of *Zara* game with two dice (so we considered $6^2 = 36$ different combinations of results). In particular, we considered the following scores:

6 = 1+5 2+4 3+3 (5 combinations out of 36)

7 = 1+6 2+5 3+4 (6 combinations out of 36)

8 = 2+6 3+5 4+4 (5 combinations out of 36)

Our aim is to evaluate pupils' approach to these non-symmetric cases.
We gave to every pupil the following test:

In *Zara* game you cast two common dice and try to guess the total score obtained. Of course some total scores are not very frequent: 12, for instance, can be obtained *only* obtaining 6 and 6 with your dice: 12 = 6+6; on the contrary, 8 can be obtained by 2+6 = 3+5 etc. Similarly it happens as regards other scores.

a) In Sara's opinion, the best choice is the total score 7, because it is the most probable score;

b) In Thomas' opinion, the best choice is (any) one of the total scores 6 and 8, because they are the most probable scores;

c) In Hugh's opinion, the best choice is (any) one of the total scores 6, 7 and 8, because they are the most probable scores;

d) In Valentine's opinion, the best choice is (any) one of the total scores 3, 4, 5, 6, 7, 8, 9, 10 and 11 (all the possible results except 2 and 12), because they are the most probable scores.

In your opinion, what is the best choice between (a), (b), (c) and (d)?

Time allowed: 20 minutes.
The results of the test (86 pupils) are given in the following table:

Answer	*Pupils*	*Percentage*
(a) best choice: 7	27	32 %
(b) best choice: 6, 8	3	3 %
(c) best choice: 6, 7, 8	51	60 %
(d) best choice: 3, 4, 5, 6, 7, 8, 9, 10, 11	2	2 %
No answer	3	3 %

Pupils were asked to justify their answers. The greater part of the pupils (60%) that preferred the answer (c) underlined that:

6 = 1+5 2+4 3+3

7 = 1+6 2+5 3+4

8 = 2+6 3+5 4+4

So they considered the same arguments proposed, more than three centuries ago, by *Gentiluomini Fiorentini*.
As regard pupils that preferred the answer (a), some of them stated that 7 is the "central" score, so it seemed to be the best choice. Only six pupils underlined, in several ways, that:

6 = 1+5 2+4 3+3 4+2 5+1

7	=	1+6	2+5	3+4	4+3	5+2	6+1
8	=	2+6	3+5	4+4	5+3	6+2	

The role of symmetry is interesting: the distance of the total score 7 from the minimum score (2) and from the maximum score (12) is the same one; similarly we can notice with reference to the scores 6 and 8 (see once again: Bagni & Cecchini, 1999).

CONCLUSIONS

Let us quote J. Fauvel and J. van Maanen:

"As every educational project, the consideration of the History of Mathematics as a component of the teaching of Mathematics implies a more or less explicit expectation of a better learning. So the research about the introduction of History of Mathematics into teaching is an important part of the research in Didactics of Mathematics" (Fauvel & van Maanen, 1997, p. 8).

According to an important conception of the Didactics of Mathematics, the main goal of the didactical research is the improvement of teaching; then it will bring to an improvement of learning. This conception brought to many remarkable results: several researchers working according to this conception gave important suggestions to improve really the teaching, by new activities, by cunnings; the presentation of some mathematical topics by historical references can be organised in this approach.

Several matters are related to the introduction of History of Mathematics in classroom practice. F. Furinghetti and A. Somaglia remember also the metacognitive sphere and write:

"We would isolate two working levels in the introduction of History into Didactics: one associated to the 'social' image of Mathematics on one hand and one rather concerning its 'inner' image on the other. The first level concerns the interventions aimed at giving motivation towards the study of Mathematics through its social (geographical, historical, commercial, linguistic) contextualization [...] The second level revives [...] the cultural dimension of Mathematics as a method, also in tight connection with the working method of other disciplines" (Furinghetti & Somaglia, 1997, p. 43).

Use of historical examples can be really interesting in informal teaching of Probability. However it caused different reactions in pupils' minds and he contributed to point out some obstacles in learning (as regards obstacles, we made reference to: Brousseau, 1983). In particular, experimental data showed that pupils' difficulties are not different from… historical difficulties that we can find in quoted arguments by *Gentiluomini Fiorentini*: so pupils' mistakes are related to difficulties dealing with non-symmetric cases and this can compared with Laplace's statements (let us remember that Laplace himself noticed: "This is one of the most difficult points of all the Theory": Laplace, 1820).

The mere proposal of an historical example, that of course is really useful in classroom practice, is not always enough to assure a full learning: the limit of this introduction consists in the uncertainty about the effects (upon the learning) of teachers' choices; effects upon the learning must be carefully verified. Of course, this does not belittle the importance of History into Mathematics Education, that is remarkable (let us remember that Piaget's opinion is well known: Piaget & Inhelder, 1975; Piaget & Garcia, 1983); but it suggests the necessity to work on the structure itself of the didactical research (Gagatsis, 1992). It is necessary to consider experimental verifies: the presence of this experimental sphere reinforces the use of History of Mathematics, changes the outlining of the research and gives it a particular epistemological status.

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