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Learning, problem solving and use of representative registers in Italian High School

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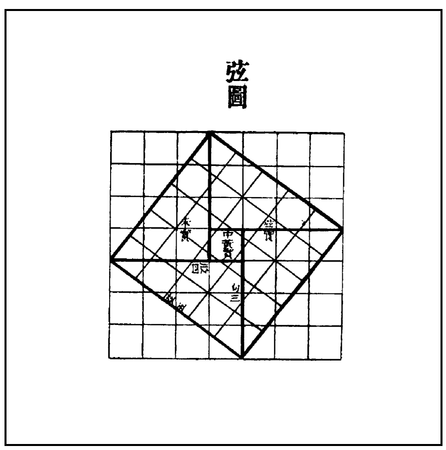
Summary. Several previous studies showed that the role of semiotic representations is very important in the learning of Mathematics: in this paper we analyse the behaviour of High School pupils (aged 17-19 years) with reference to some exercises in Trigonometry and in Analytic Geometry; an experimental research considered 196 pupils. In particular, as regard strategies and educational implications, we conclude that many pupils try to solve a problem only in the sector explicitly considered: and sometimes this is a remarkable obstacle to reach good performances and it is ineffective for the development of the ability to co-ordinate different registers of representation.

INTRODUCTION

Two sentences by pupils of Italian Secondary School will help me to introduce the subject of this work.



The visual proof of the Theorem of Pythagoras from *Chou Pei Suan Ching*

Some results given in this paper were presented to UMI-Meeting of Salsomaggiore, 2000.

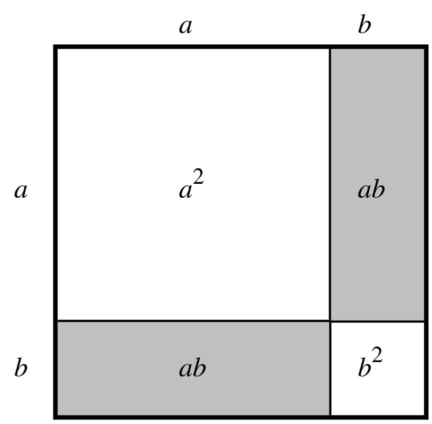
First, I am going to quote Marco, a 12 years old pupil dealing with the Theorem of Pythagoras; with reference to the statement and to the figure, he asked to me: «Well, as regards a theorem, must I state it by words or… by pictures?» I did not reply immediately to Marco, I asked his own opinion; so he stated: «It is preferable a picture: we are dealing with Geometry!»

Then I quote Anna, a pupil aged 18 years, who had to choose the best method in order to prove that $(a+b)^2 = a^2+2ab+b^2$:

1.

$$(a+b)^2 = (a+b)(a+b) = a^2+ba+ab+b^2 = a^2+2ab+b^2$$

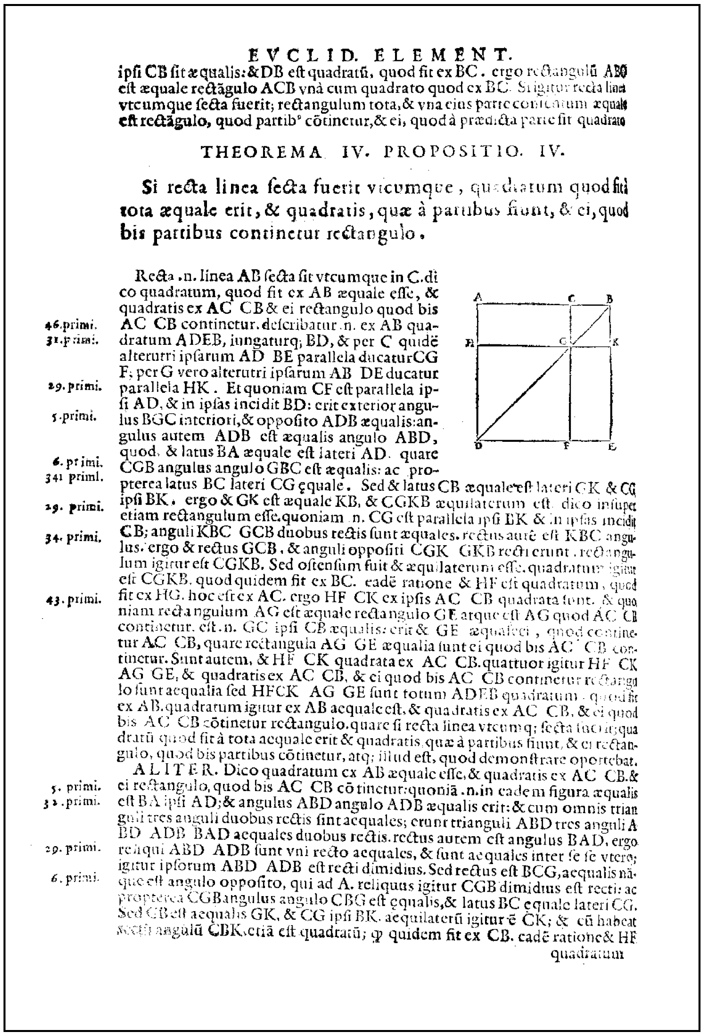
2.



Anna preferred the first method; she stated: «The algebraic method is preferable because it works upon an algebraic property only by an algebraic technique» (as regards Greek Geometric Algebra and its applications into Mathematics Education, see for instance: Kaldrimidou, 1987 and 1995; Bagni, 1998).

So both pupils seem "sectorialize" their own approaches to Mathematics. They, awarely or not, consider a particular sector of Mathematics (*Geometry*, *Algebra*); and they prefer to work only in such sector.

A first element to be underlined is the role of the visual representation (let us quote S. Vinner: «People remember visual aspects of a concept better than its analytical aspects»: Vinner, 1992, p. 212). Geometry, for instance, is explicitly based upon visual representation: didactics of Geometry and Geometric problem solving are naturally considered with reference to visual representation (Duval, 1993 and 1997; moreover: D'Amore, 1997 and chapter 5 of D'Amore, 1999).



EVCLID. ELEMENT.

ipsi CB sit æqualis: & DB est quadratũ, quod fit ex BC. ergo rectangulũ ABD est æquale rectãgulo ACB vnà cum quadrato quod ex BC. Si igitur recta linea vtcumque secta fuerit; rectangulum tota, & vna eius parte contentum æquale est rectãgulo, quod partib⁹ cõtinetur, & ei, quod à prædicta parte fit quadrato.

THEOREMA IV. PROPOSITIO. IV.

Si recta linea secta fuerit vtcumque, quadratum quod fit à tota æquale erit, & quadratis, quæ à partibus fiunt, & ei, quod bis partibus continetur rectangulo.

Recta .n. linea AB secta sit vtcumque in C. dico quadratum, quod fit ex AB æquale esse, & quadratis ex AC CB & ei rectangulo quod bis
46. primi. AC CB continetur. describatur .n. ex AB qua-
31. primi. dratum ADEB, iungaturq; BD, & per C quidẽ alterutri ipsarum AD BE parallela ducatur CG F; per G vero alterutri ipsarum AB DE ducatur
29. primi. parallela HK. Et quoniam CF est parallela ipsi AD, & in ipsas incidit BD: erit exterior angu-
5. primi. lus BGC interiori, & opposito ADB æqualis: angulus autem ADB est æqualis angulo ABD, quod & latus BA æquale est lateri AD. quare
6. primi. CGB angulus angulo GBC est æqualis: ac pro-
34. primi. pterea latus BC lateri CG ęquale. Sed & latus CB æquale est lateri GK & CG
29. primi. ipsi BK. ergo & GK est æquale KB, & CGKB æquilaterum est. dico insuper etiam rectangulum esse. quoniam .n. CG est parallela ipsi BK & in ipsas incidit
34. primi. CB; anguli KBC GCB duobus rectis sunt æquales. rectus autẽ est KBC angulus. ergo & rectus GCB, & anguli oppositi CGK GKB recti erunt. rectangulum igitur est CGKB. Sed ostensum fuit & æquilaterum esse. quadratum igitur est CGKB. quod quidem fit ex BC. eadẽ ratione & HF est quadratum, quod
43. primi. fit ex HG. hoc est ex AC. ergo HF CK ex ipsis AC CB quadrata sunt. & quoniam rectangulum AG est æquale rectangulo GE atque est AG quod AC CB continetur. est .n. GC ipsi CB æqualis: erit & GE æquale ei, quod continetur AC CB, quare rectangula AG GE æqualia sunt ei quod bis AC CB continetur. Sunt autem, & HF CK quadrata ex AC CB. quattuor igitur HF CK AG GE, & quadratis ex AC CB, & ei quod bis AC CB continetur rectangulo sunt aequalia sed HFCK AG GE sunt totum ADEB quadratum, quod fit ex AB. quadratum igitur ex AB aequale est, & quadratis ex AC CB, & ei quod bis AC CB cõtinetur rectangulo. quare si recta linea vtcumq; secta fuerit; quadratũ quod fit à tota aequale erit & quadratis, quæ à partibus fiunt, & ei rectangulo, quod bis partibus cõtinetur, atq; illud est, quod demonstrare oportebat.

ALITER. Dico quadratum ex AB æquale esse, & quadratis ex AC CB. &
5. primi. ei rectangulo, quod bis AC CB cõtinetur: quoniã .n. in eadem figura æqualis
32. primi. est BA ipsi AD; & angulus ABD angulo ADB æqualis erit: & cum omnis trianguli tres anguli duobus rectis sint aequales; erunt trianguli ABD tres anguli ABD ADB BAD aequales duobus rectis. rectus autem est angulus BAD, ergo
29. primi. reliqui ABD ADB sunt vni recto aequales, & sunt aequales inter se se vterq; igitur ipsorum ABD ADB est recti dimidius. Sed rectus est BCG, aequalis nã-
6. primi. que est angulo opposito, qui ad A. reliquus igitur CGB dimidius est recti: ac propterea CGB angulus angulo CBG est ęqualis, & latus BC equale lateri CG. Sed CB est aequalis GK, & CG ipsi BK. aequilaterũ igitur ẽ CK; & cũ habeat rectũ angulũ CBK, etiã est quadratũ; quod quidem fit ex CB. eadẽ ratione & HF

quadratum

Greek Geometric Algebra in Euclid's *Elements*
edited by Federico Commandino (1619).

A well known work by E. Fischbein is particularly devoted to visual representation of mathematical objects and to its importance into Mathematics Education (Fischbein, 1993): by his "theory of figural concepts", Fischbein states that «the integration of conceptual and figural properties in unitary mental structures, with the predominance of the conceptual constraints over the figural ones, is not a natural process. It should constitute a continuous, systematic and main preoccupation of the teacher» (Fischbein, 1993, p. 156). So if by the term "figural concept" we mean a «fusion between concept and figure» (Fischbein, 1993, p. 143), we can underline, in Fischbein' s words, that «the processes of building figural concepts in student's mind should not be considered a spontaneous effect of usual geometry courses» (Fischbein, 1993, p. 156).

The correct visual representation of a concept is a fundamental problem: the difficulty to co-ordinate different representative registers can cause sectorialization. We agree with A.H. Schoenfeld, who (dealing with different situations) writes: «Pupils are competent when they deduce and they are competent when they construct, but they often sectorialize their knowledge […] So a large sector of their knowledge remains unused and their performances in problem solving are much below to the level they could (and should) reach. An inappropriate sectorialization of activities of deduction and of activities of construction is a direct consequence of teaching» (Schoenfeld, 1986, p. 226 see moreover: Bagni, 1998).

Let us underline that the traditional mathematical curriculum, for instance as regards Italian High School, is strongly sectorialized: *Arithmetics*, *Euclidean Geometry*, *Basic Algebra*, *Analytic Geometry*, *Trigonometry*, *Calculus*, *Probability* etc. are often considered singular sectors, with their own particular rules and exercises; some of them signify… whole school-years (*Analytic Geometry*, *Trigonometry* and *Calculus* means directly 'Mathematics" in 3 rd, 4th and 5th classes of Italian *Liceo scientifico*!). Are pupils aware of this sectorialization? (Duval, 1996). What are the consequences of such situation? And are teachers aware of those consequences, for instance when they ask their pupils to connect different sectors of Mathematics?

So we considered some particular questions:

- *Question 1.* Assume that pupils' performances in some particular sectors of Mathematics are good; are their performances good as regard exercises dealing with several sectors, too?
- *Question 2.* Assume that pupils' performances related to an exercise dealing with several sectors are not good; are their performances better if the considered exercise is divided into several steps?
- *Question 3.* Is the time allowed an important element in order to obtain good performances, as regards exercises dealing with several sectors?

METHODOLOGY OF OUR WORK

We used the following tests, in order to answer to previous questions:

- *Test 1.* We considered three cards, A, B, C, in which we proposed similar exercises in Trigonometry (A), Analytic Geometry (B) and both

Trigonometry and Analytic Geometry (C). We wanted to compare pupils' performances (5 minutes allowed).

- *Test 2.* We explicitly divided exercise (card C) into two steps in order to underline the different sectors (5 minutes allowed).
- *Test 3.* We proposed once again card C: 10 minutes allowed.

We analysed pupils' behaviour with reference to four 4th High School classes (*Liceo scientifico*, 99 pupils aged 17-18 years) and four 5th classes (*Liceo scientifico*, 97 pupils aged 18-19 years), in Treviso, Italy (total 196 pupils). Their mathematical curricula were traditional; in particular, let us underline that:

- they knew main elements and techniques of Analytic Geometry; they knew Cartesian graph of $y = \sin x$ and $y = \sin^{-1}x$;
- they can solve the equation $f(x) = g(x)$ by comparison of the Cartesian diagrams of $y = f(x)$ and of $y = g(x)$;
- they knew main elements and techniques of Trigonometry.

Pupils were subdivided into five groups, A, B, C, D, E.

Test 1

We gave the following card A to every pupil belonging to groups A:

Card A

Let α and β be real numbers such that $0 \le \alpha \le \pi \wedge 0 \le \beta \le \pi$. If $\sin\alpha = \sin\beta$, find the relation connecting α and β.

We gave the following card B to every pupil belonging to groups B:

Card B

Represent in a Cartesian plan the relation expressed by:

$$y = x \quad \vee \quad y = \pi - x \qquad\qquad 0 \le x \le \pi \wedge 0 \le y \le \pi$$

We gave the following card C to every pupil belonging to groups C:

Card C

Represent in a Cartesian plan the relation expressed by:

$\sin x = \sin y$ $\quad 0 \le x \le \pi \wedge 0 \le y \le \pi$

Time: 5 minutes.
Let us summarise results of test 1 in the following tables:

Card A (39 pupils: 20 of 4th class, 19 of 5th class)			
	4^{th} class	5^{th} class	Total
$\alpha = \beta \vee \alpha = \pi - \beta$	16 (80%)	11 (58%)	27 (69%)
Only $\alpha = \beta$	3 (15%)	5 (26%)	8 (21%)
Other answ. or no answ.	1 (5%)	3 (16%)	4 (10%)

Card B (40 pupils: 20 of 4th class, 20 of 5th class)			
	4^{th} class	5^{th} class	Total
Correct diagram	17 (85%)	14 (70%)	31 (78%)
Only segm. of $y = x$	1 (5%)	0 (0%)	1 (2%)
Other answ. or no answ.	2 (10%)	6 (30%)	8 (20%)

Card C (39 pupils: 20 of 4th class, 19 of 5th class)			
	4^{th} class	5^{th} class	Total
Correct diagram	4 (20%)	5 (26%)	9 (23%)
Only segm. of $y = x$	3 (15%)	6 (32%)	9 (23%)
Other answ. or no answ.	12 (65%)	8 (42%)	21 (54%)

Clearly we can underline a really remarkable difference between pupils' performances related to cards A and B (69% and 78%) and to card C (only 23%).

Test 2

We gave the following card to every pupil belonging to groups D:

Card D

Let α and β be real numbers such that $0\le\alpha\le\pi \wedge 0\le\beta\le\pi$. If $\sin\alpha = \sin\beta$, find the relation connecting α and β.

Then represent in a Cartesian plan the relation expressed by:

$\sin x = \sin y$ $0\le x\le\pi \wedge 0\le y\le\pi$

Time: 5 minutes.

Let us summarise results (we shall compare them with results of card C):

Card D (39 pupils: 20 of 4th class, 19 of 5th class)			
	4th class	5th class	Total
Correct diagram	12 (60%)	9 (47%)	21 (54%)
Only segm. of $y = x$	4 (20%)	3 (16%)	7 (18%)
Other answ. or no answ.	4 (20%)	7 (37%)	11 (28%)

Card C (39 pupils: 20 of 4th class, 19 of 5th class)			
	4th class	5th class	Total
Correct diagram	4 (20%)	5 (26%)	9 (23%)
Only segm. of $y = x$	3 (15%)	6 (32%)	9 (23%)
Other answ. or no answ.	12 (65%)	8 (42%)	21 (54%)

So pupils' performances are higher as r egards card D (54%; card C: 23%).

Test 3

We gave the card C to every pupil belonging to groups E, 10 minutes allowed.

Let us summarise results of test 3 in the following tables (we shall compare results of card C with 10 minutes allowed and results of card C, 5 minutes):

Card C (time: 10 minutes) (38 pupils: 19 of 4th class, 19 of 5th class)			
	4th class	5th class	Total
Correct diagram	3 (16%)	7 (37%)	10 (26%)
Only segm. of $y = x$	6 (31%)	7 (37%)	13 (34%)
Other answ. or no answ.	10 (53%)	5 (26%)	15 (40%)

Card C (time: 5 minutes) *(39 pupils: 20 of 4th class, 19 of 5th class)*			
	4th class	5th class	Total
Correct diagram	4 (20%)	5 (26%)	9 (23%)
Only segm. of $y = x$	3 (15%)	6 (32%)	9 (23%)
Other answ. or no answ.	12 (65%)	8 (42%)	21 (54%)

In this case, as regard performances, we cannot point out great differences.

DISCUSSION

Let us notice first that our tests considered a small number of pupils (from the statistic point of view, moreover, we did not consider a particular sample): so, in order to avoid over-interpretations, it would be necessary to investigate students' conceptions by further tests, administered to many students.

However, we pointed out a trend (with reference to the considered exercise):

- *Answer 1.* If pupils' performances in some particular sectors are good, we *cannot* assume that their performances are good as regard exercises dealing with several sectors of Mathematics, too.
- *Answer 3.* Time allowed is *not* an important element in order to obtain good performances, as regards exercises dealing with several sectors.

So (in examined situations; further researches can be planned) many pupils show hesitancy, perplexity when they are going to solve exercises dealing with different sectors and different representative registers. In Schoenfeld's words (previously quoted), we notice a real, dangerous *sectorialization*. Moreover, it is important to underline that pupils' performances are clearly better if the considered exercise is divided into its steps:

- *Answer 2.* Pupils' performances are better if the considered exercise is divided into several steps, each of them dealing with a particular sector.

So we can state that the considered obstacle is an educational one: the sector implicitly or explicitly suggested by the text evokes techniques to be used by solvers (and, for instance, representative registers employed); now we can notice the influence of clauses of the didactical contract: in fact pupils works only in the considered sector ([1]).

References

Bagni, G.T.: 1998, Visualization and Didactics of Mathematics in High School: an experimental research, *Scientia Paedagogica Experimentalis*, XXXV, 1, 161-180.

D'Amore, B.: 1997, Matite – Orettole – Przxetqzyw. Le immagini mentali dei testi delle situazioni-problema influenzano davvero la risoluzione?, *L'insegnamento della Matematica e delle Scienze integrate*, 20A, 3, 241-256.

D'Amore, B.: 1999, *Elementi di Didattica della Matematica*, Pitagora, Bologna.

Duval, R.: 1993, Registres de répresentation sémiotique et fonctionnement cognitif de la pensée, *Annales de Didactique et de Sciences Cognitives*, 5, IREM, Strasbourg.

Duval, R.: 1994, Les différents fonctionnements d' une figure dans une démarche géométrique, *Repres IREM*, 17, october.

Duval, R.: 1996, Quale cognitivo per la didattica della matematica?, *La Matematica e la sua didattica*, 3, 250-269.

Duval, R.: 1997, La compréhension des énoncés de problème de mathématisation: de la lecture à la résolution, D' Amore, B. & Gagatsis, A. (Eds.), *Didactics of Mathematics-Technology in Education*, Erasmus ICP-96-G-2011/11, 25-46, Thessaloniki.

Fischbein, E.: 1993, The theory of figural concepts, *Educational Studies in Mathematics*, 24, 139-162.

Gagatsis, A. & Dimarakis, I.: 1996, The limit concept; Difficulties-obstacles of Students' Understanding, Gagatsis, A. & Rogers, L., *Didactics and History of Mathematics*, Erasmus, Thessaloniki.

Kaldrimidou, M.: 1987, *Images mentales et représentations en Mathématiques chez les mathématiciens et les étudiants en Mathématiques*, Thèse 3éme cycle, Université Paris 7, Paris.

Kaldrimidou, M.: 1995, Lo status della visualizzazione presso gli studenti e gli insegnanti di Matematica, *La Matematica e la sua didattica*, 2, 181-194.

Poli, P. & Zan, R.: 1996, Il ruolo delle convinzioni nella risoluzione di problemi, *La Matematica e la sua didattica*, 4, 440-466.

Schoenfeld, A.H.: 1986, On having and using geometric knowledge, Hiebert, J. (Ed.), *Conceptual and procedural knowledge: the case of Mathematics*, 225-263, Erlbaum, Hillsdale.

Vinner, S.: 1992, Function concept as prototype for problems in Mathematics. Harel, G. & Dubinsky, E. (Ed.), *The concept of Function: aspects of Epistemology and Pedagogy*, MAA Notes, 25, 195-213.

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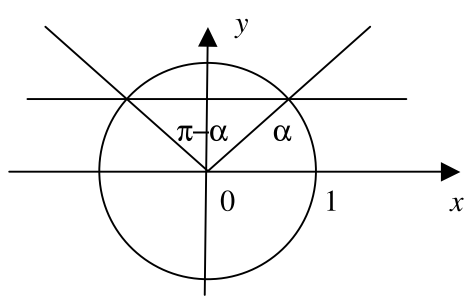
(1) As regards the role of different representative registers, the trigonometric resolution (card A) mainly deals with the algebraic representative register; let us underline that many pupils giving the correct answer mentioned the traditional following "diagram":



Of course this situation is referred to the graphic representative register; so we can notice a remarkable co-ordination of employed registers (algebraic and graphic: Duval, 1994). So co-ordination of different representative registers cannot be based only upon pupils' skill, but upon educational choices and behaviours, too. Affective elements can be considered really important (as regards a different school-level, see for instance: Poli & Zan, 1996, pp. 454-455): the fact itself to work into a particular sector of Mathematics is sometimes implicitly considered as a reassuring situation by pupils and suggests strategies and choices; as previuosly pointed out, this can deeply influence pupils' performances.