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Integral and Continuity In High School students' conceptions

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Summary. As regards the concept of continuity of the set of the real numbers, the idea of integral in the learning of mathematics is investigated, referred to Italian High School (*Liceo scientifico*, 5th class). The status of these concepts is studied by two tests, in which Dirichlet's function, Riemann's integral and Lebesgue's integral, referred to Peano-Jordan measure, are proposed. We conclude that the traditional study of the Calculus in High School does not allow the full knowledge of the concepts of continuity of the set of the real numbers and of integral.

1. Historical remarks

Historically, the settlement of the set of real numbers and of the concept of infinite set took place by several classical researches; in particular, in XIX century, works by Dedekind, by Weierstrass and by Cantor were fundamental (Kline, 1972, II). Julius W. R. Dedekind (1831-1916) wrote in 1872 the important work entitled *Continuity and irrational numbers*; his name is associated to:

Dedekind's Axiom. If we consider a segment AB divided in two parts S, T (not empty) such that: every point of the segment belongs to one of the two parts; every point of the part S precedes (from A to B) every point of T, then there is only one point C of AB, belonging to the part S or to the part T (but not to both of them) such that every point of AB preceding C belongs to S and every point of AB following C belongs to T.

Karl T. W. Weierstrass (1815-1897), too, gave an axiom about continuity:

Weierstrass' Axiom. Every indefinite succession of point of a segment A, B, ..., P,... arranged according to one of the verses of the segment, tends to a limit.

The axiom about continuity by Georg Cantor (1845-1918) is fundamental:

Cantor's Axiom. If we consider the sets S and T of segments such that: any segment belonging to S is not greater than every segment belonging to T; there are a segment belonging to T and a segment belonging to S such that their difference is lower than any given segment ε, then there is a segment that is not lower than any segment belonging to S and that is not greater than any segment belonging to T ([1]).

Georg Cantor introduced the set theory, that reached great importance in didactics of mathematics, too; Cantor's research es about infinite sets are fundamental ([2]). Let us remember, in Cantor's words, the concept of actual infinity that backed up and then replaced the ancient potential infinity (by Aristotle):

"Mathematical infinity... is crescent beyond every limit or indefinitely decrescent, and it is a quantity that remains *finite*. I call it *improper infinity*. Moreover, recently, another kind of infinity... took place... By that... the infinity is considered as concentrated in a certain point. When infinity occours in this form, I call it *proper infinity*" (in: Bottazzini-Freguglia-Toti Rigatelli, 1992, p. 428; in this paper translations are ours).

Cantor introduced moreover the concept of power of a set, not only for finite sets, but for infinite sets too, by $\aleph_i$ numbers: one of main achievements by Cantor was the possibility to "compare" and "classify" infinite sets: for example, he distinguished the power $\aleph_0$ for the set **Q** of real rational numbers, that is dense in the set **R** of real numbers, and the power $\aleph_1$ for the whole set **R** ([3]).

For a long time Cantor's ideas about infinite sets were considered rather difficult. So Cantor's $\aleph_i$ numbers are not included into traditional mathematical curricula of High School (particularly referred to Italian *Liceo scientifico*); but, as we shall see, this absence causes some problems in students' conceptions, most of all about continuity (Tall, 1980; D'Amore, 1996).

The introduction of the notion of integral is an interesting moment, as regards the study of the concepts of density and of continuity. Let us give an historical note, referred to XIX century (see moreover the historical preface in: Gagatsis-Dimarakis, 1996):

"Cauchy succeeded in proving the existence of an integral for every continuous function and he defined the integral when the function has a leap or it becomes infinite, too. But when the Calculus developed, the necessity to consider integrals of functions whose behaviour is irregular was pressing. The problem of the functions that can be integrated was re-examined by Riemann in his work of 1854 about trigonometric series" (Kline, 1972, II, p. 1119).

So M. Kline remembers that G. F. Bernhard Riemann (1826-1866) underlined the necessity to give full details of the modern concept of integral;

he gave the conditions so that a function can be integrated with reference of sums today named integral sums ([4]): such integral is named Riemann's integral.

Functions that can be integrated and functions that cannot be integrated (according to Riemann) are important in didactics of mathematics. In fact it is well-known that in some particular cases of discontinuity of the integrand, Riemann's integral can be inapplicable.

Let us see an example of a function that cannot be integrated according to Riemann (we shall use the function χ_A: $\mathbf{R}\to\mathbf{R}$ referred to the set $A\subseteq\mathbf{R}$, that is 1 if and only if $x\in A$, and 0 if and only if $x\in\mathbf{R}\backslash A$). Let $\mathbf{Q}$ be the set of the rational real numbers and let us consider the function given by: $y = \chi_{[a;b]}\chi_{\mathbf{Q}}$ ($a\in\mathbf{R}$, $b\in\mathbf{R}$, $a<b$), defined in $\mathbf{R}$; it is different from 0 if and only if $x\in\{x\in\mathbf{R}: a\le x\le b \wedge x\in\mathbf{Q}\}$. The function $x\to\chi_{[a;b]}\chi_{\mathbf{Q}}$ cannot be integrated according to Riemann in $[a; b]$ for every $a\in\mathbf{R}$, $b\in\mathbf{R}$, $a<b$. This function is sometimes named *Dirichlet's function*, from Peter Gustav Lejeune Dirichlet (1805-1859) ([5]).

The concept of integral can be extended such that Dirichlet's function and functions similar to Dirichlet's function can be integrated. The study of the integral had, in the first years of XX century, a strong impulse by Emile Borel (1871-1956) and by Henri Lebesgue (1875-1941); Lebesgue achieved a new settlement of the concepts of measure and of integral ([6]).

The introduction of Lebesgue's measure is not the crucial point of our study. Integral according to Lebesgue can be referred to the Lebesgue's measure (it is referred to the functions that can be integrated according to Lebesgue), but to the traditional Peano-Jordan's measure of a segment, too (by that, the measure of $[a; b]\subseteq\mathbf{R}$ is $b-a$): of course, in such case, the integral is referred only to the functions that can be integrated according to Riemann and it coincides (with regard to its result and to the concept) with the "old" integral according to Riemann.

2. Lebesgue's integral and Peano-Jordan's measure: an experimental research

The introduction of Lebesgue's integral was historically a very important step: the integral (and the concept of function that can be integrated) missed its absolute uniqueness and *the fundamental role of the measure* was clearly underlined. Of course the growth from Riemann's integral to Lebesgue's integral does not depend only upon the different way of going about its definition (the partition of the interval *of the range* into correspondance with the interval in which the integral is considered), but it is strongly based upon the fundamental evolution from Peano-Jordan's measure to Lebesgue's measure.

In short, if we apply Lebesgue's method with reference to the "old" Peano - Jordan's measure, we get the "old" Riemann's integral.

We want to analyze an important matter: what is the difference (from the didactic point of view) between the introduction of Riemann's integral and the introduction of Lebesgue's integral (with Peano -Jordan measure)?

And in particular: what of these methods is advantageous in the didactics of mathematics in High School?

Our work is based upon the following question: is it possible to replace the traditional introduction of the Riemann's integral (by the partition of the interval in which the integral is considered) with the introduction of Lebesgue's integral (by the partition of the interval of the range into correspondance with the interval in which the integral is considered), with reference to Peano-Jordan's measure?

Our research is based upon the following tests:

- **Test 1.** We examinated the status of the concept of function that can be integrated (according to Riemann) in a class in which Riemann's integral was introduced in the traditional way.

- **Test 2.** We repeated the previous test in a class in which Lebesgue's integral was introduced (referred to Peano-Jordan's measure).

In our work, the analysis of students' behaviour in High School, particularly referred to Italian *Liceo scientifico*, considered:

- A 5th class of *Liceo scientifico* (pupils aged 18-19 years), in Treviso, Italy, total 25 students; they knew the concepts of limit, of continuous function, of derivative and Riemann's integral. They knew the χ_A function of a set A. They did not know Lebesgue's integral.

- A 5th class of *Liceo scientifico* (pupils aged 18-19 years), in Treviso, Italy, total 24 students; they knew the concepts of limit, of continuous function, of derivative and the integral according to Lebesgue's introduction, but with reference to Peano-Jordan's measure (it was applied only to the functions that can be integrated according to Riemann). Moreover they knew the χ_A function of a set A.

Students of both classes were asked to answer to the questions of the following test (time: 20 minutes; no textbooks or calculators allowed):

Say if the following functions $\mathbf{R}\to\mathbf{R}$ *can be integrated in* [0; 2]*:*

(a) $y = x^2+3x+2$

(b) $y = \chi_{[0;1[} + 2\chi_{[1;2]}$

(c) $y = \chi_{[0;1]} + 2\chi_{[1;2]}$

(d) $y = \chi_{[0;2]}\chi_{\mathbf{Q}}$

3. Test 1: results

	The function can be integrated		The function cannot be integrated		No answer	
(a)	24	96 %	0	0 %	1	4 %
(b)	15	60 %	4	16 %	6	24 %
(c)	13	52 %	7	28 %	5	20 %
(d)	10	40 %	8	32 %	7	28 %

Almost all students stated that the function (a) can be integrated. Some difficulties arose about funcions (b) and (c) that are not continuous at $x = 1$.

Several answers about the function (d) are disappointing: only 32 % of the students stated that this function cannot be integrated according to Riemann in [0; 2]; 40 % stated that this function can be integrated according to Riemann and 28 % did not answer.

Students justified their answers in some interviews.

A remarkable part of the students that stated that the function (b) and (c) cannot be integrated (function (b): 3 out of 4; function (c): 4 out of 7) stated that because of the presence of a point ($x = 1$) at which these functions are not continuous. For example:

"A continuous function can be integrated; these functions are not continuous at 1, so they cannot be integrated" (Umberto).

This mistake is clearly logical: a sufficient condition is interpreted as a necessary condition.

The following justification is interesting (the student stated that the function (b) can be integrated and that the function (c) cannot be integrated):

"Function (c) is 1 from 0 to 1 (except the point 1 itself), it is 3 at $x = 1$ and it is 2 from 1 (except the point 1 itself) to 2. I thought that it is strange that this function can be integrated: its integral would be equal to the integral of the function (c), but the point at $x = 1$ is... higher! Now I understand that I was wrong: the point (1; 3) of the function (c), by oneself, does not bound a real surface so the integral cannot be different with respect to the function (b)" (Marco).

We shall consider again this important justification later.

About the function (d), the greater part of the students (8 out of 10) that stated that it can be integrated wrote that the area bounded by its Cartesian graph is not different from the area of the rectangle whose base is 2 (from [0; 2]) and whose altitude is 1. For example:

"Between two numbers there are infinite rational numbers. S o I thought that the part of the plane underneath the graph of the function (d) is all the rectangle, made by an infinite number of small columns infinitely thick" (Andrea).

So once more *the difference between dense and continuous is not clear in students' conceptions*.

Several students that stated that the function (d) cannot be integrated did not give any justification; let us remember the following statement:

"The function (d) does not bound a surface whose area can be computed because its graph is not a full segment, but it is just a thick succession of points" (Sergio).

The comment of a pupil is interesting: she underlined that it is difficult to decide if a function can be integrated or not because of the necessity to refer to a long procedure (the partition of the considered interval, the computation of the integral sums...) and not just to a definition, as the traditional definition of continuous function:

"I had several problems about the functions that can be integrated and the function that cannot be integrated. For example, it is easy to say if a function is continuous or if its derivative exists: I have clear rules, I must calculate some limits, there are not many exceptions. While about the functions to be integrated, I do not know what I must look for..." (Francesca).

The identification of the functions that can be integrated (according to Riemann) is a remarkable problem for many students of High School (surely harder than the identification of continuous functions or of the functions having derivative, as Francesca noted). Frequently the students are looking for an easy rule, that can be applied in exercises, and several of them thought that the functions that can be integrated are (only!) the continuous functions.

Once more the difference between dense and continuous is not well comprised by several students, even though they know many fundamental topics of the Calculus. The results about Dirichlet's function (d) underline that the difference between

$\{x\in \mathbf{R}: 0\le x\le 2 \wedge x\in \mathbf{Q}\}$ (dense and that cannot be measured by Peano-Jordan's measure)

and

$\{x\in \mathbf{R}: 0\le x\le 2\}$ (continuous, misurabile by Peano-Jordan's measure)

is felt just by few students.

By the following test we shall verify that the approach of the integral according to Lebesgue (but only with reference to Peano-Jordan's measure, so referred to the functions that can be integrated according to Riemann) improves the comprehension of the characteristics of some functions, while it does not change the situation about other functions.

4. Test 2: results

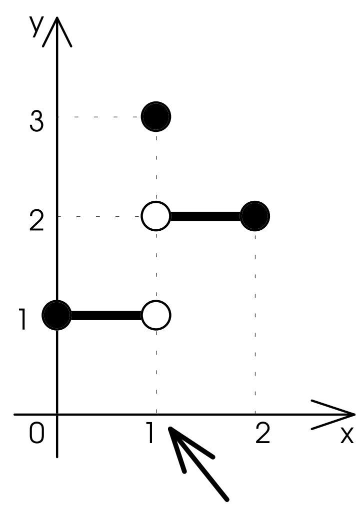
	The function can be integrated		The function cannot be integrated		No answer	
(a)	22	92 %	1	4 %	1	4 %
(b)	20	83 %	3	13 %	1	4 %
(c)	18	75 %	2	8 %	4	17 %
(d)	7	29 %	9	37 %	8	33 %

As for test 1, almost all students stated that the function (a) can be integrated. As regards the functions (b) and (c) (that are not continuous at x = 1) results of test 2 are better than results of test 1: correct answers are 83 % (b) (test 1: 60 %) and 75 % (c) (test 1: 52 %).

The situation about function (d) is still rather bad: only 37 % of the students stated that it cannot be integrated according to Riemann in [0; 2] and this is not much better than 32 % (test 1); 29 % of the students stated that this function can be integrated (test 3: 40 %) and 33 % gave no answers (test 1: 28 %).

Students justified their answers in some interviews.

Some of the students that stated that the functions (b) and (c) cannot be integrated gave this answer because of the presence of a point (x = 1) at which these functions are not continuous. For example, let us see the justification given by Stefano, who draws the following picture (that is the correct Cartesian graph of the function (c)):



'I thought that the points of the functions (b) and (c) at which they are not continuous cause any problems, but I was wrong. For example, the graph of the function (c) can be divided into three parts, referred to y = 1, to y = 2 and to y = 3. The part referred to 1 comes from [0; 1[, that is 1; the part referred to 2 comes from]1; 2], that is 2; the part referred to 3 comes from x = 1, so from a single point, whose measure is 0, because it is: 1−1. So I can find the total area; I have: 1 ·1+2·1+3·0, so I have 3" (Stefano).

About the function (d), many students that stated that it can be integrated (9 out of 15) showed difficulties about the measure (by Peano-Jordan's measure) of $\{x \in \mathbf{R}: 0 \le x \le 2 \wedge x \in \mathbf{Q}\}$. Let us see a justification:

"I reach $y = 1$ from the rational numbers between 0 and 2. Their measure is $2-0 = 2$, so the area is $2 \cdot 1 = 2$" (Tullio).

So dense and continuous are once more confused.

About the students that stated that the function (d) cannot be integrated, let us see the following justification:

"I had not a real surface to be measured, because it is always interrupted by a lot of little stripes that I should remove. Rational and irrational numbers are mixed ant they cannot be divided. I think that the area cannot be computed, so the function (d) cannot be integrated" (Antonella).

The introduction of the integral according with Lebesgue's method (with reference to Peano-Jordan's measure) seems to improve the comprehension of the notion of the functions that can be integrated according to Riemann. Data above given about functions (b) and (c) are clear and Stefano's justification (referred to the measure of the set whose elements x are such that $f(x) = 3$) is surely convincing, better than Marco's one (just referred to the absence of a "real surface" in correspondence with $x = 1$).

About Dirichlet's function, results of test 1 and of test 2 are similar: doubts and difficulties about the distinction between dense and continuous remain, so the absence of a specific introduction of infinity and of Cantor's numbers $\aleph_i$ leads to the impossibility to characterize clearly the functions that can be integrated according to Riemann.

5. Conclusions

In our opinion a didactic role for the integral according to Lebesgue is important and it is to be hoped for. In fact, the traditional introduction of Riemann's integral (that is by partition of the interval belonging to axis x) can be replaced by the introduction of Lebesgue's integral (that is by partition of the interval belonging to axis y). This choice is not very hard and it can be with reference to Peano-Jordan's measure: the students would learn a very important method (Lebesgue's integral is frequently applied in modern mathematics); moreover the students would distinguish integral's role from measure's role; they would realize that the integral itself (that, as above noted, does not change if we consider Riemann's integral and Lebesgue's integral with reference to Peano-Jordan's measure) strictly depends upon the chosen measure ([7]).

Previous remarks *would be particularly important in presence of a specific introduction of infinity and of Cantor's numbers* $\aleph_i$. In fact some examples of functions that can be integrated according to Riemann (with $f(x)$ different from

0 only at x belonging to sets that cannot be measured by Peano-Jordan's measure) are connected to the concepts of dense and of continuous set (8).

Notes

(1) Several mathematicians between XIX and XX century made researches about discrete and continuous; in particular we remember Charles Méray (1835-1911) and Hermann Weyl (1885-1955). See for example: Bagni, 1996, II.

(2) Surely Cantor found in Bernhard Bolzano (1781-1848) a source of cues about actual infinity. U. Bottazzini notes: "Distinction between actual infinity and potential infinity was suggested by Bolzano, too, in his *Paradoxien des Unendlichen*, a work held in high regard by Cantor" (Bottazzini, 1990, p. 252).

(3) Only in the last period of Cantor's life the full correctness of his ideas was accepted by mathematicians (Kline, 1972, II, p. 1172). C.B. Boyer writes: "Cantor's personal tragedy is comf orted by praises of one of the most important mathematicians in the first part of our century, David Hilbert, who... exclaimed: «No one will expel us from the paradise created by Cantor for us»" (Boyer, 1968, p. 655).

(4) About Riemann: Laugwitz, 1995. About the first half of XX century: Pier, 1994.

(5) Some Authors (for example Prodi, 1970, p. 308) define this function only in [0; 1]; some Authors refer the definition to the function χ_Q of the set of the rational reals.

(6) The class of the sets that can be measured according to Lebesgue is very large; a set that cannot be measured according to Lebesgue was found by Giuseppe Vitali (1875-1932). About Lebesgue's integral: Kolmogorov-Fomin, 1980 and Rudin, 1966.

(7) About Lebesgue's measure and about measure's theory, see the important (and difficult) work: Halmos, 1994; about measure and probabilities, see: Doob, 1994 and Simonnet, 1996.

(8) Let's finally remember that the difference between dense sets and continuous sets cannot be correctly visualized: R. Duval notices that learning by graphic representations "needs a particular work" and "it is impossible to rely their use on spontaneous interpretation of pictures and of images" (Duval, 1994). The "double nature", ideal, abstract and on the other hand real, of several mathematical objects, according to the theory of figural concepts by E. Fischbein (Fischbein, 1993), does not seem relevant in this case (see moreover: Bagni, 1997).

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