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Some “impossible” problems in High School students’ behaviour

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Summary. In this paper, the behaviour of High School students is investigated, with reference to some “impossible” problems (in particular we examined Italian *Liceo scientifico*, students aged 17-19 years). We conclude that some cases of “impossibility” are improperly “extended” to similar exercises: the influence of didactic and experimental contracts is remarkable in students’ behaviour.

1. INTRODUCTION: IMPOSSIBLE PROBLEMS (1)

Students’ behaviour in order to solve some “impossible” problems is frequently very interesting (a clear classification of “impossible” problems can be found in: D’Amore & Sandri, 1993; see moreover: Sfard, 1991, p. 2) (2). As regards pupils aged 10-11 years, some effects of experimental and didactic contracts in elaboration of responses to “impossible” problems are deeply studied in: Schubauer Leoni & Ntamakiliro (1994). In that work, the Authors analysed some reasoning strategies used by students, in particular with reference to interactions between “public” and “private” aspects of answer formulations. In fact, when students deal with a problem without solution, many of them are brought to elaborate their responses with reference to main aspects of questions. So M.L. Schubauer Leoni and L. Ntamakiliro underlined the primary importance of both experimental contract and didactic contract in order to explain students’ line of conduct (see Schubauer Leoni & Ntamakiliro, 1994, p. 94; moreover: Chevallard, 1988 and Schubauer Leoni, 1988).

In this paper, we shall consider some “impossible” problems; let us underline that we shall deal with “impossible” problems (as indicated in: D’Amore & Sandri, 1993, paragraph 3-B, pp. 344-345), but not with “absurd” problems (as indicated in: D’Amore & Sandri, 1993, paragraph 3-A, p. 344; see for example: Baruk, 1985): so we shall examine if this difference can be relevant in students strategies and if it can be connected with "public" and "private" aspects of students' reasoning. So this matter could be considered as a (partial) answer to the question proposed by B. D'Amore and P. Sandri in the quoted paper: can different kinds of "impossible" problems cause different students' behaviours? (D'Amore & Sandri, 1993, p. 346).

We shall examine some aspects of students' behaviour with particular reference to High School students (in particular to Italian *Liceo scientifico*, students aged 17-19 years). First of all, let us present a common mistake related to limits (students aged 18-19 years) (3).

2. LIMITS: A COMMON MISTAKE

The greater part of students knows that:

$$\lim_{x\to+\infty} \sin x$$

does not exist. The function $x \to \sin x$ is a periodic one, and it is well-known that if $x \to f(x)$ is a periodic function, $\lim_{x\to+\infty} f(x)$ exists if and only if $x \to f(x)$ is a constant function (4).

Let us notice that $\lim_{x\to+\infty} \sin x$ is an "impossible" exercise, but it is not an absurd one: in other words, the (correct) resolution of this exercise consists of the justification of its impossibility.

Moreover, let us notice that also the limit:

$$\lim_{x\to+\infty} x\sin x$$

does not exist. In this case, as we shall see, the "result" of the mentioned limit can be (improperly) related to the "result" of the previous one; however it is very important to underline that the function $x \to x\sin x$ is *not* a periodic one; so it is quite correct to state that $\lim_{x\to+\infty} x\sin x$ does not exist, but it is clearly wrong to say that it does not exist because the function $x \to x\sin x$ just... "contains" a periodic function.

As we shall see, sometimes these examples are improperly "extended" and they bring to a common mistake; in fact, several students state that:

$$\lim_{x\to+\infty} \frac{1}{x}\sin x$$

does not exist; and of course this is false. In our opinion, the "presence" itself of $\sin x$ "in" the function $x \to \frac{1}{x}\sin x$ brings some student to relate $\lim_{x\to+\infty} \frac{1}{x}\sin x$ to the (impossible) limits $\lim_{x\to+\infty} \sin x$ and $\lim_{x\to+\infty} x\sin x$.

In this paper, we shall analyse some mistakes, with reference to High School students (in particular to Italian *Liceo scientifico*, students aged 17-19 years). We shall present:

- an experimental research about limits (students aged 18-19 years).
- an experimental research about trigonometric equations (students aged 17-19 years).
- an experimental research about algebra and trigonometry (students aged 17-18 years).

3. LIMITS: AN EXPERIMENTAL RESEARCH

3.1. Method of tests

A test was proposed to students belonging to four 5th classes of a *Liceo scientifico* (High School; pupils aged 18-19 years) in Treviso, Italy, total 94 students; their curricula were standard: they knew basic elements of the theory of limits (in particular, they knew that some limits do not exist) and of trigonometry.

The following test was proposed to the pupils:

Calculate:		
(a) $\lim_{x\to+\infty} \sin x$	(b) $\lim_{x\to+\infty} x\sin x$	(c) $\lim_{x\to+\infty} \frac{1}{x}\sin x$

Time: 6 minutes (we wanted that students examine the problem 'at a glance'). No textbooks or electronic calculators allowed.

By this test we wanted to examine the influence of the first and of the second resolutions in the interpretation of the last one.

3.2. Results of test and considerations about results

(a)	$\lim_{x\to+\infty} \sin x = 0$	14	15%
	$\lim_{x\to+\infty} \sin x = +\infty$	2	2%
	$\lim_{x\to+\infty} \sin x = \pm\infty$	1	1%
	$\lim_{x\to+\infty} \sin x = 1$	1	1%
	$\lim_{x\to+\infty} \sin x = \pm 1$	1	1%
	$\lim_{x\to+\infty} \sin x$ does not exist	68	73%
	no answer	7	7%

(b)	$\lim_{x\to+\infty} x\sin x = 0$	1	2%
	$\lim_{x\to+\infty} x\sin x = +\infty$	6	6%
	$\lim_{x\to+\infty} x\sin x = \pm\infty$	3	3%
	$\lim_{x\to+\infty} x\sin x$ does not exist	59	63%
	no answer	25	26%
(c)	$\lim_{x\to+\infty} \frac{1}{x}\sin x = 0$	24	26%
	$\lim_{x\to+\infty} \frac{1}{x}\sin x = +\infty$	1	1%
	$\lim_{x\to+\infty} \frac{1}{x}\sin x = 1$	1	1%
	$\lim_{x\to+\infty} \frac{1}{x}\sin x$ does not exist	49	52%
	no answer	19	20%

We can notice that the first limit is (correctly) considered as an 'impossible" exercise by 73% of students; the second limit is considered 'impossible" by 63%. As it is well-known, the third limit is not 'impossible" (only 26% of students stated that it can be calculate and it is 0); but 52% of students fell in this mistake.

We suppose that the cause of the mentioned mistake is the 'presence" of the function $x\to\sin x$ in the function $x\to\frac{1}{x}\sin x$; moreover, why did many students state that $\lim_{x\to+\infty} x\sin x$ does not exist? Was this (correct) statement referred just to the 'presence" of the function $x\to\sin x$ in the function $x\to x\sin x$?

3.3. Justifications given by students

Some students gave interesting justifications (interviews took place in the classroom, in other pupils' presence); as regards students that stated that $\lim_{x\to+\infty} x\sin x$ does not exist, the greater part of them (48 out of 59) just noticed that $\lim_{x\to+\infty} \sin x$, too, does not exist. Of course this justification cannot be accepted: this shows that several students have problems with the limit concept.

For example:

«Of course, I know that $\lim_{x\to+\infty} \sin x$ does not exist: our teacher told us clearly that $\lim_{x\to+\infty} \sin x$, $\lim_{x\to+\infty} \cos x$, $\lim_{x\to+\infty} \mathrm{tg}x$ etc. do not exist: in fact, for example, $\sin x$

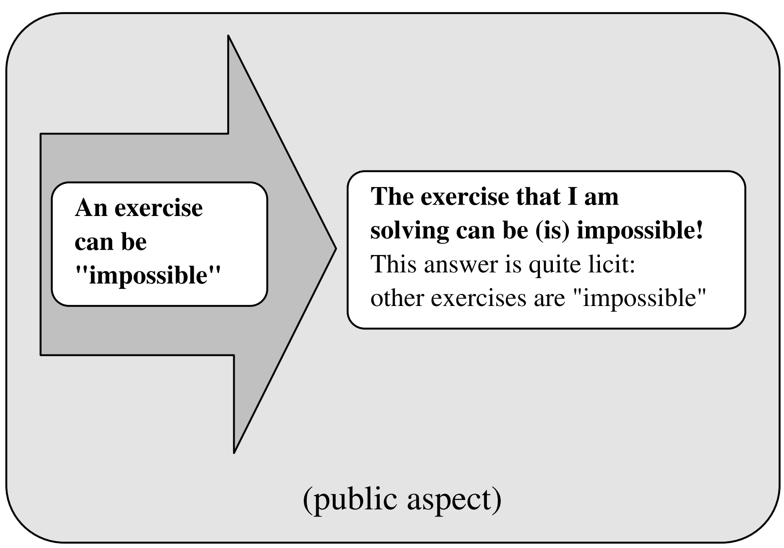
can be 1, or 0, or −1, and so on. So $\lim_{x\to+\infty}\sin x$ is impossible, it cannot be calculated: the Cartesian graph of $y = \sin x$ does not approach a single number, when x grows higher and higher. Of course, this happens for $\lim_{x\to+\infty} x\sin x$, too» (Dino).

Let us notice that in Dino's answer the emphasis is always on $\lim_{x\to+\infty}\sin x$; only in the final part of his answer he just mentioned $\lim_{x\to+\infty} x\sin x$.

As regards $\lim_{x\to+\infty}\frac{1}{x}\sin x$, too, almost all students that stated that it does not exist (43 out of 49) just noticed that $\lim_{x\to+\infty}\sin x$ does not exist (justifications are quite similar to Davide's one). In this case, this wrong justification brought many students to a mistake.

We must underline that many students seem clearly aware of the general situation: they know that some limits exist and other limits do not exist. As regards "public" aspect (according to: Schubauer Leoni & Ntamakiliro , 1994), students did not show any embarrassment or bewilderment to state that exercises like $\lim_{x\to+\infty}\sin x$ or $\lim_{x\to+\infty} x\sin x$ are "impossible", that is, to say that these limits do not exist: so, as regards limits, they know that the result "impossible" is an accepted one (let us notice that a similar "public" statement is frequently refused by some clauses of the didactic contract: an exercise *must* have a result; see for example: Baruk, 1985; Micol, 1991; Bagni, 1997).

We can summarize the situation by the following picture:



4. TRIGONOMETRIC EQUATIONS: AN EXPERIMENTAL RESEARCH

4.1. Method of tests

The results of the test above presented show that several students improperly "extended" the impossibility of $\lim_{x\to+\infty}\sin x$ to $\lim_{x\to+\infty}\frac{1}{x}\sin x$. Is this a "casual" mistake? Moreover, is it just referred to limits? We wanted to examine students' behaviour with reference to another topic from the mathematical curriculum of High School.

So a test was proposed to students belonging to two 4th classes and two 5th classes of a *Liceo scientifico* (High School; pupils aged 17-19 years) in Treviso, Italy, total 90 students (two 4th classes: 23 and 24 pupils respectively; two 5th classes: 23 and 20 pupils respectively); their curricula were standard: in particular, they knew basic elements of trigonometry.

First of all, the following card (A) was proposed to the pupils:

> *Card A*
>
> Read the following resolution:
>
> $$(2\sec x-1)\cos x(\cos x-1) = 0$$
>
> $$\sec x = \frac{1}{2} \quad \vee \quad \cos x = 0 \quad \vee \quad \cos x = 1$$
>
> Let us underline that the presence of sec*x* requests: $x \neq \frac{\pi}{2} + k\pi \quad k\in\mathbf{Z}$.
>
> The first equality is impossible: we know that sec*x* cannot be $\frac{1}{2}$.
>
> From the second equation we have:
>
> $$x = \frac{\pi}{2} + k\pi \quad \text{being:} \quad k\in\mathbf{Z}$$
>
> but **we cannot accept these solutions**, because we put: $x \neq \frac{\pi}{2} + k\pi \quad k\in\mathbf{Z}$.
>
> From the third equation we have:
>
> $$x = \pi+2k\pi \quad \text{being:} \quad k\in\mathbf{Z}$$
>
> So the complete solutions of the equation $(2\sec x-1)\cos x(\cos x-1) = 0$ are:
>
> $$x = \pi+2k\pi \quad \text{being:} \quad k\in\mathbf{Z}$$

Five minutes later, the following card (B) was proposed to the pupils:

Card B

Read the following resolution and answer:

$(\cos x - \sin x)\cos x = 0$

$\cos x - \sin x = 0 \quad \vee \quad \cos x = 0$

The first equation can be solved as it follows:

$\cos x - \sin x = 0 \quad \Rightarrow \quad 1 - \tan x = 0 \quad$ being: $\quad x \neq \frac{\pi}{2} + k\pi \quad k \in \mathbf{Z}$

so we have:

$\tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4} + k\pi \quad$ being: $\quad k \in \mathbf{Z}$

From the latter equation we have:

$x = \frac{\pi}{2} + k\pi \quad$ being: $\quad k \in \mathbf{Z}$

What are, in your opinion, the complete solutions of the equation $(\cos x - \sin x)\cos x = 0$?

Write your answer:

o $\quad x = \frac{\pi}{2} + k\pi \quad \vee \quad x = \frac{\pi}{4} + k\pi \quad$ being: $\quad k \in \mathbf{Z}$

o $\quad x = \frac{\pi}{4} + k\pi \quad$ being: $\quad k \in \mathbf{Z}$

Time (as regards card B): 3 minutes (we wanted that students examine the problem 'at a glance'). No textbooks or electronic calculators allowed.

By this test we wanted to examine the influence of the first resolution (card A) in the interpretation of the latter one (card B): in particular, as regards the first resolution, $x = \frac{\pi}{2} + k\pi$ cannot be accepted (and this impossibility is clearly expressed); well, what are students' opinions as regards the latter one?

4.2. Results of test and considerations about results

$x = \frac{\pi}{2} + k\pi \quad \vee \quad x = \frac{\pi}{4} + k\pi \quad$ being: $\quad k \in \mathbf{Z}$	29	32%
$x = \frac{\pi}{4} + k\pi \quad$ being: $\quad k \in \mathbf{Z}$	51	57%
no answer	10	11%

So the greater part of students improperly refused $x = \frac{\pi}{2} + k\pi$; could this mistake be referred to the similar (correct, of course) choice in the first equation (card A)?

Now we must realize if (and how) the resolution presented in the card A really influenced some answers to the question in the card B.

4.3. Justifications given by students

Interviews took place in the classroom, in other pupils' presence.

As regards students that answered that $x = \frac{\pi}{2} + k\pi$ cannot be accepted, let us remember the following justification:

«I examined the resolution presented in card A: when I deal with sec$x$, or with tan$x$, I am immediately forced to consider the condition $x \neq \frac{\pi}{2} + k\pi$. I was wrong, now I understand that there is a difference between the situation ins card A and in card B» (Paolo, 4^{th} class); 32 justifications are similar to this one.

The following statement is very interesting:

«First of all, I considered the resolution in the card A: why was this resolution shown us? I thought that I had to guess something from it, and surely that resolution was a correct one: so I thought that probably I had to point out a resolution similar to the resolution in the card A and I chose $x \neq \frac{\pi}{2} + k\pi$» (Giulio, 5^{th} class).

Giulio's justification is really quite clear: he considered the card A and he tried to guess what he had to do, or, better, what the author of the test wanted him to do: clearly this behaviour can be referred to the experimental contract.

5. ALGEBRA AND TRIGONOMETRY: AN EXPERIMENTAL RESEARCH

5.1. Method of tests

Two tests were proposed to students belonging to two 4^{th} classes and two 5^{th} classes of a *Liceo scientifico* (High School; pupils aged 17-19 years) in Treviso, Italy, total 97 students (two 4^{th} classes: 25 and 23 pupils respectively; two 5^{th} classes: 25 and 24 pupils respectively); we shall identify them by group A (the first 4^{th} and 5^{th} classes, total 50 puplis) and group B (the second 4^{th} and 5^{th} classes, total 47 puplis); all students had the same mathematics teacher; their curricula were standard: they knew basic elements of trigonometry; in particular, they knew the fundamental trigonometric equality: $\sin^2 x+\cos^2 x = 1$ and the equalities $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}$ (being $-1\leq x\leq 1$) and $\tan^{-1}x+\cot^{-1}x = \frac{\pi}{2}$.

The first test (A) was proposed to the 50 pupils of the group A (4th class: 25 pupils; 5th class: 25 pupils):

Test A

1) Calculate:

$\sin^2 x+\cos^2 x+\sqrt{\cos x-3}$ = ...

2) Calculate:

$\sin^2 x+\cos^2 x+(\sin^2 x+\cos^2 x-1)\sqrt{\cos x-3}$ = ...

3) Draw the Cartesian graph of:

$y = \sin^{-1}x+\cos^{-1}x+\tan^{-1}x+\cot^{-1}x$

Time: 9 minutes (we wanted that students examine the problem 'at a glance'). No textbooks or electronic calculators allowed.

The second test (B) was proposed to the 47 pupils of the group B (4th class: 23 pupils; 5th class: 24 pupils):

Test B

1) Remember that $\cos x<3$ for every $x\in\mathbf{R}$. Calculate:

$\sin^2 x+\cos^2 x+\sqrt{\cos x-3}$ = ...

2) Remember that $\cos x<3$ for every $x\in\mathbf{R}$. Calculate:

$\sin^2 x+\cos^2 x+(\sin^2 x+\cos^2 x-1)\sqrt{\cos x-3}$ = ...

3) You know that $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}$ (being $-1\leq x\leq 1$) and $\tan^{-1}x+\cot^{-1}x = \frac{\pi}{2}$.
Draw the Cartesian graph of:

$y = \sin^{-1}x+\cos^{-1}x+\tan^{-1}x+\cot^{-1}x$

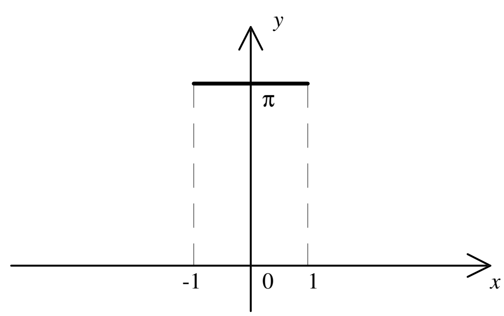
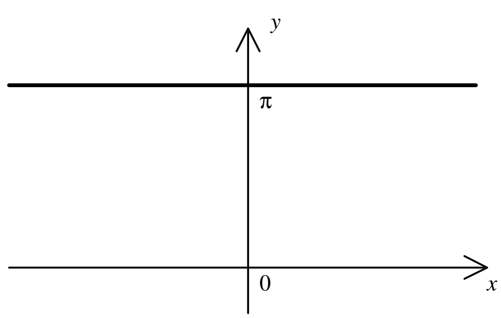
Time: 9 minutes. No textbooks or electronic calculators allowed.

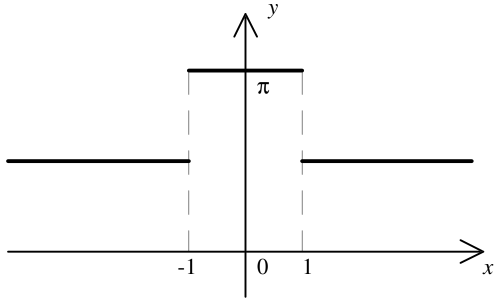
Let us notice that the test A and the test B are quite similar, but in the latter one there are more informations than in the first one (exercises 1 and 2: $\cos x<3$ for every $x\in\mathbf{R}$; as regards exercise 3, in particular, it is important to remember: $-1\leq x\leq 1$).

It is well-known that different representations of a problem are important as regards students' behaviour in problem solving (Fischbein, Tirosh & Hess, 1979; Silver, 1986; Arcavi, Tirosh & Nachmias, 1989; Tsamir & Tirosh, 1997):

so we wanted to point out the influence of these further informations in students’ answers.

5.2. Results of test and considerations about results

	Test A (50 pupils)		*Test B* (47 pupils)	
Exercise 1				
‘impossible”	33	66%	28	60%
1	13	26%	8	17%
no answer	14	28%	11	23%
Exercise 2				
‘impossible”	12	24%	16	34%
1	29	58%	20	43%
no answer	9	18%	11	23%
Exercise 3				
	16	32%	19	40%
	9	18%	3	7%

	7	14%	11	23%
other (wrong) graphs or no answer	18	36%	14	30%

These results are somehow amazing: in spite of the considerable difference between the informations given, the differences between the results of test A and the results of test B are rather slight.

5.3. Justifications given by students

Interviews took place in the classroom, in other pupils' presence.

First of all, as regards students that gave no answer to exercises 1 and 2 (tests A and B), we wanted to realize if they considered this exercise as an "impossible" one. Let us consider the following justification:

«I did not answer because I was not sure: of course, I know that $\sin^2x+\cos^2x$ is 1; but I realized that in the root there is something wrong. I was not sure about it, so I did not give an answer» (Roberto, 4th class).

Almost all the justifications are similar to this one. We can conclude that we cannot consider these answers as the answer "impossible".

As regards students that answered "1" (exercise 1), let us remember the following justification:

«Well, I know that $\sin^2x+\cos^2x = 1$ and that $\sqrt{\cos x - 3}$ cannot be calculated. Then I wrote that $\sin^2x+\cos^2x+\sqrt{\cos x - 3}$ is 1» (Giovanni, 4th class, test A).

«The root $\sqrt{\cos x - 3}$ does not exist because $\cos x$ is always lower than 3; so I have only $\sin^2x+\cos^2x$, that is 1» (Mario, 5th class, test B).

So, in some students' mind, when a quantity cannot be calculated it is "nothing" and it can be considered... 0.

As regards students that answered "1" (exercise 2), let us remember the following justification:

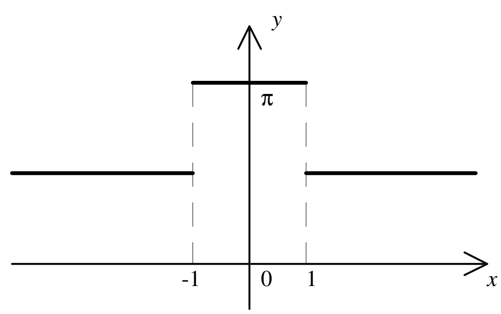
«They told me that $\cos x<3$ for every $x\in\mathbf{R}$: of course, it is well-known! Moreover, I know that $\sin^2x+\cos^2x = 1$, so $\sin^2x+\cos^2x-1$ is 0 and the final result is 1+0 = 0. There is a law that states that anything multiplied by 0 is always 0» (Isabella, 4th class, test B).

Isabella stated that "anything multiplied by 0 is always 0": of course, this would be correct if "anything" is replaced by "any number". As regards exercise 1, let us notice that Isabella's answer was "impossible". So she considered that $\sqrt{\cos x - 3}$ (alone) cannot be calculated, while $0 \cdot \sqrt{\cos x - 3}$ is 0.

As regards exercise 3, the following justification is very interesting:

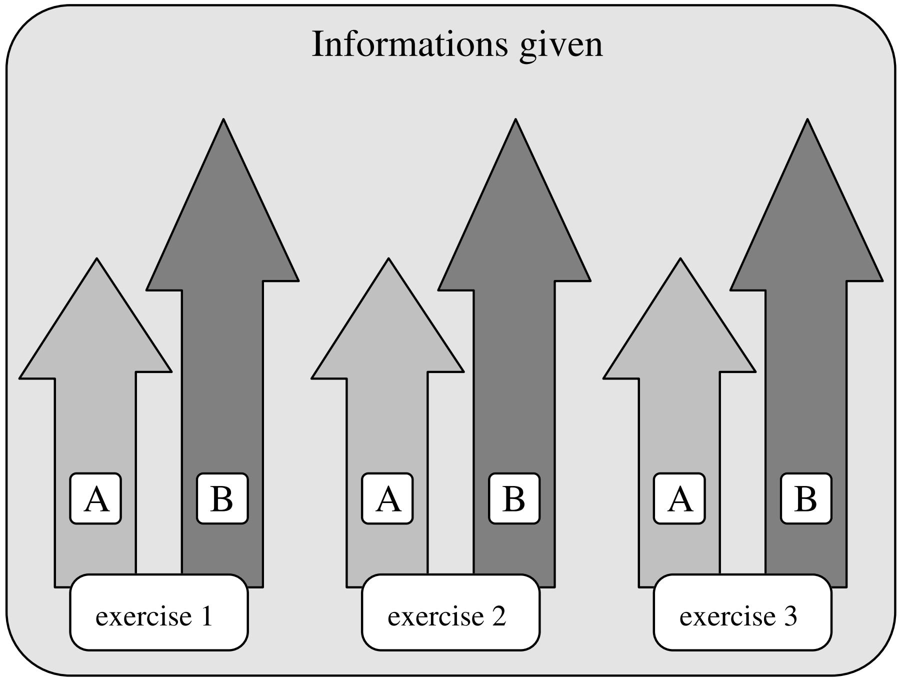
«It is impossible that a test is so simple... What should I have to do? In the card I can read that $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}$ and that $\tan^{-1}x+\cot^{-1}x = \frac{\pi}{2}$, so clearly I have $\sin^{-1}x+\cos^{-1}x+\tan^{-1}x+\cot^{-1}x = \pi$; well, should I draw just a straight line? What is the meaning of this exercise?» (Mirko, 4th class, test B).

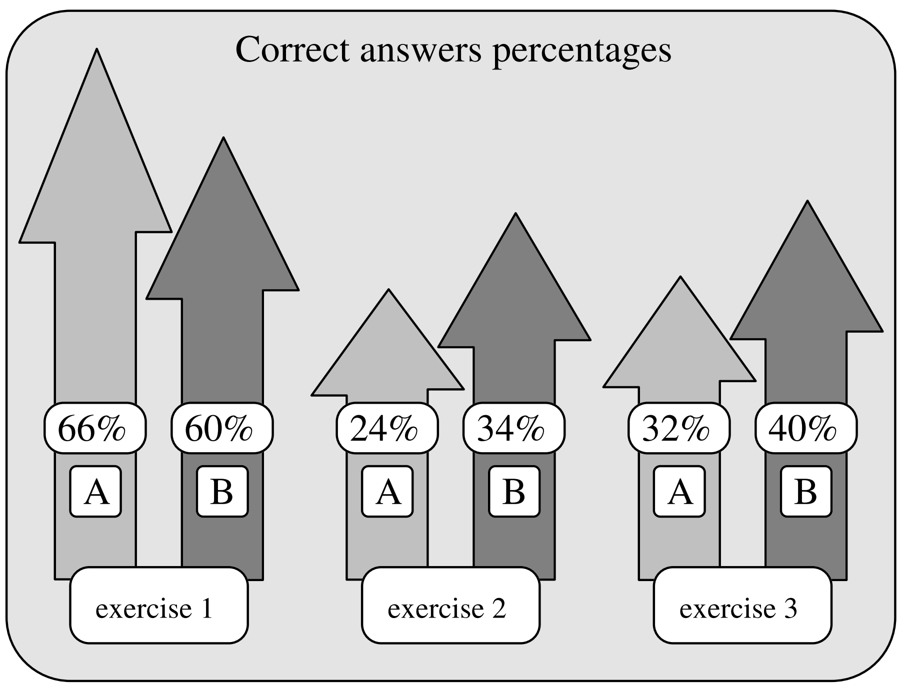
So Mirko was looking for the "meaning" of the exercise: as we can see, the influence of experimental contract is rather clear. As regards exercise 3, Mirko's answer was:



It seems that Mirko forced himself to find... something to do: "What is the meaning of this exercise?" According experimental contract, an exercise must be interesting, or somehow difficult. In other words, an exercise (and its answer) must have always "a meaning": Mirko did not find interesting enough the exercise 3 (and its answer), so he seems to ask himself: well, $\frac{\pi}{2}+\frac{\pi}{2} = \pi$, this is clear, and then? "Should I draw just a straight line?" Too easy to be correct...

As previously noticed, in spite of the considerable difference between the informations given, the differences between the correct answers of test A and the correct answers of test B are slight: as regards exercise 1, the correct answers percentage referred to the test A (66%) is (just a little) lower than the correct answers percentage referred to the test B (60%).


Informations given
A
B
A
B
A
B
exercise 1
exercise 2
exercise 3


Correct answers percentages
66%
60%
24%
34%
32%
40%
A
B
A
B
A
B
exercise 1
exercise 2
exercise 3

Above pictures are just qualitative representations: let us underline once again that the differences between the correct answers of test A and the correct answers of test B are really slight.

6. GENERAL CONCLUSIONS

Results of the experimental researches previously presented needs some remarks. The situations previously described show that the influence of both didactic contract and experimental contracts is interesting: especially as regards "public" aspect (according to: Schubauer Leoni & Ntamakiliro, 1994), the presence itself of an "impossible" exercise seems to reassure some students about the possibility and the correctness of the answer "impossible": we could say that this possibility is considered nearly as a new clause of the didactic contract, that is going to replace the clause by which every exercise must have a result (one and only one: as regards some "impossible" trigonometric exercises under the influence of the didactic contract, see: Bagni, 1997).

It seems that the role of examples (and counterexamples) is important to make students aware of uncorrect answers and of their conflicting ideas ([5]). Nevertheless, the use of examples (and counterexamples) is not conclusive: let us remember once again that the experimental contract induces some students to refer "impossible" examples to a lot of cases, without particular controls (and it would be interesting to consider this behaviour from an affective point of view, too) ([6]): of course this situation can cause dangerous mistakes.

So how can we overcome these effects of the didactic contract and, especially, of the experimental contract?

We could think that increased informations make it easier to deal with "impossible" exercises (see paragraph 5.1). But experimental results above given (see paragraph 5.2) show clearly that this is not always true; moreover, sometimes we find mistakes in easy exercises because in students' opinion they are "too easy" (remember, for example, Mirko's justification, in paragraph 5.3). Some students seem to think that the difficulty of different exercises must be always approximately the same: if an exercise is remarkably "too easy", surely... there is something underneath, because it is impossible that an experimental research is based upon "too easy" exercises.

From this point of view, in our opinion, it is strictly necessary that students are made aware of the possibility of different levels of difficulty: this would be important both as regards the didactic contract and as regards the experimental contract.

We conclude that the deep influence of both didactic contract and experimental contracts must be considered as a main contribution to reasoning strategies used by students (in several school-levels): so this must be carefully taken into account by researchers in mathematics education.

NOTES

(1) Some results proposed in the prsent work were published in *Progetto Alice*, 2, 2000.

(2) For example, in a recent work about situations related to problems with a missing datum, with reference to pupils aged 8-9 years and 12-13 years, the Authors noticed that many pupils "imagine" the missing datum in order to be able to solve the considered problem (D'Amore & Sandri, 1998).

(3) As regards the learning of the notion of limit, and in particular some important misconceptions, see for example: Cornu, 1980; Davis & Vinner, 1986; Dimarakis & Gagatsis, 1986. In our opinion, however, these misconceptions would not influence directly the described situation.

(4) As regards $\lim_{x \to -\infty} f(x)$, of course, the same situation can be pointed out.

(5) As regards the presence of conflicting answers and of ideas that are incompatible with each other, see for example: Tall, 1990; Tsamir & Tirosh, 1992 and 1997; let us remember that several researches showed that sometimes students do not realize the presence of conflicting answers: Stavy & Berkovitz, 1980; Hart, 1981; and sometimes the presence of ideas that are incompatible with each other is not considered completely illicit, forbidden; see: Schoenfeld, 1985; Tirosh, 1990.

(6) We do not think that the obstacles previously examined can be considered as epistemological ones or (only) as educational ones (see for example the fundamental classification in: Brousseau, 1983; Vergnaud, 1989, pp. 168-169). If we consider them as educational obstacles, we must underline that the influence of affective aspect is surely remakable (D'Amore & Martini, 1997). Then, in our opinion, they should be regarded as *affective obstacles*, too: so it is difficult to overcome them completely just by educational means, like for example showing of counterexamples.

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