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Didactics of Infinity: Euclid's proof and Eratosthenes' sieve
Prime numbers and potential infinity in High School

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Summary. In this paper the idea of infinity in the learning of mathematics in classroom practice is investigated, referred to Italian High School (*Liceo scientifico*, pupils aged 16-19 years). A brief historical preface is given, mentioning the contraposition of potential infinity and actual infinity. Then the status of some infinite concepts is studied by two tests, about Euclid's proof and about Eratosthenes' sieve. We conclude that infinity is introduced in the sense of potential infinity and that the traditional study of the Calculus in High School does not allow the full knowledge of the concept of (actual) infinity.

1. INTRODUCTION

Historically, the concepts of potential infinity and of actual infinity are very ancient. In fact Aristotle (384-322 b.C.) distinguished potential infinity and actual infinity ([1]): mathematical infinity, in Aristotle's opinion, is always potential: he, avoiding paradoxes (for example, famous Zeno's paradoxes [Arrigo-D'Amore, 1992, pp. 29-34]), strongly refused actual infinity.

Aristotle's resolution influenced for a long time the idea of infinity. B. D'Amore notes:

«Aristotle's prohibition to the mathematicians to use actual infinity must be considered like a real dogma... Several Authors, in the Middle Ages and in the Renaissance, but later, too, analyzed the idea of actual infinity... But Aristotle's heavy heritage was forever present» [Arrigo-D'Amore, 1992, p. 41].

([1]) L. Geymonat writes: «A variable quantity is a "potential infinity" if, although taking finite values, it can grow higher than every point; if for example we imagine to divide a segment by repeated halvings... the number of the parts, finite, can grow higher than every point. "Actual infinity" is referred to a set, really formed by an infinite number of elements; if for example we imagine to divide a segment in all its points... it is an actual infinity, because there is not a number measuring the whole quantity of these points» [Geymonat, 1970, I, p. 58].

The opposition between potential infinity and actual infinity was evident after Calculus' birth: let us resume the status of researches about Calculus' foundation in XVIII century by M. Kline's words:

«In XVIII century... the difference among a very big number and an "infinity" was neglected and it seemed self-evident that a theorem true for every *n* was true for *n* infinite, too. In a similar way, an incremental ratio was replaced by a derivative and the sum of a finite number of terms hardly ever was distinguished from an integral» [Kline, 1991, I, p. 506].

In XIX century, Georg Cantor's researches about infinite sets are very important ([2]). Let us remember, in Cantor's words, the concept of actual infinity that backed up and then replaced the ancient potential infinity:

"Mathematical infinity... is crescent beyond every limit or indefinitely decrescent, and it is a quantity that remains *finite*. I call it *improper infinity*. Moreover, recently, another kind of infinity... took place... By that... the infinity is considered as concentrated in a certain point. When infinity occours in this form, I call it *proper infinity*" (in [Bottazzini -Freguglia-Toti Rigatelli, 1992, p. 428]).

For a long time Cantor's fundamental ideas about infinite sets were considered rather difficult ([3]). So Cantor's $\aleph_i$ numbers are not included in the traditional mathematical curriculum of High School (particularly referred to Italian *Liceo scientifico*); but, as we shall see, this absence may cause some problems in students' conceptions.

2. STRUCTURE AND METHOD OF OUR RESEARCH

Many Authors wrote, in the last years, about didactics of the infinity (more than 350 titles are listed in [D'Amore, 1996]; see for example the very important works [Duval, 1983], [Tall, 1980] and [Waldegg, 1993]).

([2]) Surely Cantor found in Bernhard Bolzano (1781-1848) a source of cues about actual infinity. U. Bottazzini notes: "Distinction between actual infinity and potential infinity was suggested by Bolzano, too, in his *Paradoxien des Unendlichen*, a work held in high regard by Cantor" [Bottazzini, 1990, p. 252].

([3]) Only in the last period of Cantor's life the correctness of his ideas was accepted [Kline, 1991, II, p. 1172]. C.B. Boyer writes: "Cantor's personal tragedy is comforted by praises of one of the most important mathematicians in the first part of our century, David Hilbert, who... exclaimed: «No one will expel us from the paradise created by Cantor for us»" [Boyer, 1982, p. 655] .

The opposition between potential infinity and actual infinity is reflected in didactics of mathematics ([4]); in particular, the intuitive efficacy of potential infinity, as a quantity that can be progressively and indefinitely increased, can make preponderant the role of this idea in comparision with the concept, mathematically exacting, of actual infinity (for example, some difficulties connected to actual infinity are studied in [Tsamir-Tirosh, 1992]).

Our aim was therefore to examinate some conceptions of the students of High School about infinity. So our work analyzed students' approach to infinity. In particular, we examinated two moments of the traditional mathematical curriculum of High School (mainly referred to Italian *Liceo scientifico*):

- the introduction, in 3rd class of *Liceo scientifico* (pupils aged 16-17 years) of the infinite set of prime numbers, with Euclid's proof;
- the settlement of the concept of infinity, in 5th class of *Liceo scientifico* (pupils aged 18-19 years), by the concept of limit; in this situation, students' ideas were investigated by a test based upon Eratosthenes' sieve.

Let us show our work by the following picture:

Our work:	
1. Infinity in 3rd class of *Liceo scientifico* (16-17 years)	**2. Infinity in 5th class of *Liceo scientifico* (18-19 years)**
Test 1 (about Euclid's proof of the infinity of the set of prime numbers)	*Test 2* (about Eratosthenes' sieve)
3. Conclusions	
Infinity in the traditional curriculum of *Liceo scientifico*.	

3. TEST 1: HOW MANY PRIME NUMBERS ARE THERE?

3.1. METHOD OF TEST 1

Test 1 was based upon Euclid's proof of the proposition that states the infinity of the set of prime numbers.

([4]) B. D'Amore writes: "Disci plines in which infinity is potential are above all Analysis, Geometry, Arithmetic; those in which infinity is studied as actual are still Analysis, Arithmetic, Geometry and Logic; I should remember Chaos Theory and the study of Fractals, too..." [D'Amore, 1996].

The famous XX proposition of Ninth Book of Euclidean *Elements* is referred to the (potential) infinity of the set of prime numbers [Hardy-Wright, 1938]:

Prime numbers are always more than any quantity of prime numbers considered [Euclid, 1970].

Euclid's proof is the following, according to P. Ribenboim:

'Suppose that $p_1 = 2 < p_2 = 3 < ... < p_r$ are... primes. Let $P = p_1 \cdot p_2 \cdot ... \cdot p_r + 1$ and let p be a prime dividing P; then p cannot be any of the $p_1, p_2, ..., p_r$, otherwise p would divide the difference $P - p_1 \cdot p_2 \cdot ... \cdot p_r = 1$, which is impossible. So this prime p is still another prime, and $p_1, p_2, ..., p_r$ would not be all the primes" [Ribenboim, 1980, p. 3].

Of course Euclid's approach to infinity is clearly related to Aristotle's opinion, so, as previously noted, it is related to potential infinity.

The following test was proposed to students belonging to a 3rd class of a *Liceo scientifico* (High School) in Treviso, Italy, total 24 students (their mathematical curricula were standard; they knew the concepts of set and its simbology; a specific preparation of infinite sets, with the definition of infinite set and with $\aleph_i$ numbers, were not proposed; they knew the Euclid's propositions and its proof, above given):

Consider Euclid's proposition (XX, Ninth Book of *Elements*):

Prime numbers are always more than
any quantity of prime numbers considered.

1) What can you say about the set P of prime numbers, according to this famous proposition?

2) Are the following sentences true or false?

a) The greatest prime number does not exist.
b) For every $n \in \mathbf{N}$ it is possible to find a prime p such that $p > n$.
c) The set P of prime numbers is infinite.

3) What of the above sentences (a)-(b)-(c) would you chose to express exactly Euclid's proposition? (Chose only one proposition).

Time: 15 minutes.

3.2. RESULTS OF TEST 1

Answers to question (1):

The set P is an infinite set	22 (92 %)
No answer	2 (8 %)

Answers to question (2):

	True	False	No answer
(a) The greatest prime number does not exist	21 (88 %)	1 (4 %)	2 (8 %)
(b) For every $n \in \mathbf{N}$ it is possible to find a prime p such that $p>n$	21 (88 %)	0 (0 %)	3 (12 %)
(c) The set P of prime numbers is infinite	22 (92 %)	0 (0 %)	2 (8 %)

Answers to question (3):

(a) The greatest prime number does not exist	6 (25 %)
(b) For every $n \in \mathbf{N}$ it is possible to find a prime p such that $p>n$	3 (13 %)
(c) The set P of prime numbers is infinite	15 (62 %)
No answer	0 (0 %)

3.3. TEST 1: CONSIDERATIONS ABOUT RESULTS

Euclid's proposition (and of course Euclid's proof), as above underlined, is referred to potential infinity; the greater part of the students considered it as a correct introduction of infinity: 15 students out of 24 (62 %) expressed the Euclidean proposition by the statement 'The set P of prime numbers is infinite" (so referred to a set which is considered actually infinite) and only 9 students out of 24 (38 %) expressed the Euclidean proposition by other statements, related to potential infinity.

3.4. TEST 1: JUSTIFICATIONS GIVEN BY STUDENTS

Some students gave interesting justifications. For example:

"By Euclid's proposition I know that I can find primes greater and greater, so that there exist infinitely many primes. I mean this by saying that the set P of prime numbers is infinite" (Antonella).

"I preferred the sentence (b) because it expresses mathematically the infinity of the set of prime numbers" (Giovanni).

Giovanni's justification (looking for a... "mathematical" expression) can be related to a clause of the *didactical contract* [Brousseau, 1987] called by B. D'Amore and P. Sandri "e. g. f." ("esigenza della giustificazione formale", necessity of formal justification) [D'Amore-Sandri, 1996]; its presence can be found already in the Primary School and it becomes progressively binding in the Secondary School and in the High School.

3.5. TEST 1: CONCLUSIONS

- Potential infinity was plainly accepted by the students of 3rd class of Italian *Liceo scientifico* (High School); the learning of the concept of infinity, in this stage of the curriculum can be related to a potential conception.
- Several students preferred expressions involving formal justifications, according to a clause of *didactical contract*.

4. TEST 2: ERATOSTHENES' SIEVE

4.1. METHOD OF TEST 2

The test 2 was based upon Eratosthenes' sieve [Ribenboim, 1980], that can be expressed in the following way. We shall write:

$$M(k)=\{n\in \mathbf{N}: \quad n=mk \wedge m\in \mathbf{N}\} \qquad \begin{cases} C_0=\mathbf{N}\setminus\{0;1\} \\ P_0=\varnothing \\ C_{i+1}=C_i\setminus M(\min C_i) \\ P_{i+1}=P_i\cup\{\min C_i\} \end{cases}$$

By repeating indefinitely this procedure, the set P of primes is given by:

$$\mathrm{P}=\lim_{i\to+\infty} P_i$$

(and Euclid's proof, above remembered, shows that P_i is a non-constant sequence for $i>k$, for every $k\in\mathbf{N}$).

The following test was proposed to students belonging to a 5th class of a *Liceo scientifico* (High School) in Treviso, Italy, total 23 students (their mathematical curricula were standard; they knew the concepts of limit; a specific preparation of infinite sets, with the definition of infinite set and with $\aleph_i$ numbers, was not proposed; they did not know Euclid's proposition about infinity of the set of prime numbers):

> Consider the following sets:
>
> $$M(k)=\{n\in\mathbf{N}:\quad n=mk\wedge m\in\mathbf{N}\}\qquad\begin{cases}C_0=\mathbf{N}\setminus\{0;1\}\\P_0=\varnothing\\C_{i+1}=C_i\setminus M(\min C_i)\\P_{i+1}=P_i\cup\{\min C_i\}\end{cases}$$
>
> The set P = $\lim\limits_{i\to+\infty}P_i$ is the set of prime numbers.
>
> 1) What can you state about the number of the elements belonging to P, according to the introduction of the set P above given?
>
> 2) Justify your answer to the question (1).

Time: 15 minutes.

4.2. RESULTS OF TEST 2

1) Answers:

The elements belonging to the set P are infinitely many	23 (100 %)
No answer	0 (0 %)

2) Justifications:

In the limit P = $\lim\limits_{i\to+\infty}P_i$, i tends to $+\infty$	16 (70 %)
Prime numbers are infinite because natural numbers are infinite	3 (13 %)

No answer 4 (17 %)

4.3. TEST 2: CONSIDERATIONS ABOUT RESULTS

Although the introduction of the set P of prime numbers based upon Eratosthenes' sieve is not directly related to the infinity of P, all the students stated that P is an infinite set; several students justified that by the presence of a limit (they underlined that in the limit P = $\lim_{i\to+\infty} P_i$, i tends to $+\infty$).

4.4. TEST 2: JUSTIFICATIONS GIVEN BY STUDENTS

Some students gave interesting justifications. For example:

"The set P is an infinite set because it is introduced by an infinite proceeding, as we can see from the considered limit, P = $\lim_{i\to+\infty} P_i$. This limit would be clearly unuseful and quite not necessary for the introduction of a finite set" (Christian and some other students).

So Christian and several other students did not consider the possibility of the limit (being $i\to+\infty$) of a constant function...

"I think that P is infinite because all prime numbers belong to the set of natural numbers and the set of natural numbers, too, is an infinite set. Prime numbers are disorderly placed, and they are placed in every part of the set of natural numbers. So primes are clearly infinitely many" (Pamela).

Of course, Pamela tacitally assumed that the set P is an infinite subset of the (infinite) set of natural numbers.

4.5. TEST 2: CONCLUSIONS

- As above noted, the formal introduction of the set P of prime numbers by the limit P = $\lim_{i\to+\infty} P_i$ induced many students to state that P is an infinite set. They did not know Euclid's proposition about the infinity of the set of prime numbers, but they believed that the use of a limit being $i\to+\infty$ is enough to affirm that the set P is infinite.
- Some students stated that P is an infinite set because prime numbers are "disorderly placed" in the infinite set of natural numbers. This statement is clearly based upon the tacital assumption of the infinity of P itself: this confirms that several students have real problems talking about infinity.

5. GENERAL CONCLUSIONS

The introduction of the concept of infinity in High School is really an important stage of the mathematical curriculum. Learning of this fundamental concept must be carefully watched by the teacher, to avoid misunderstandings.

In particular, the study of the Calculus (5th class) does not improve the situation: results of students of 5th class and their justifications are not very good (about the traditional intuitive introduction of the concept of limit in the High School, mainly referred to the potential infinity, see for example [Bagni, 1996]).

The traditional mathematical curriculum of High School, and particularly the mathematical curriculum of Italian *Liceo scientifico*, can be strongly improved. The absence of a specific introduction of infinite sets (and in particular of the numbers $\aleph_i$) troubles some important matters. In particular these didactic lacks strongly limit a real improvement of the didactics of the concept of actual infinity: so a correct introduction of infinite sets and of $\aleph_i$ numbers would be indispensable for the didactics of mathematics in High School.

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