In Proceedings of CERME–5. Larnaca 2007, 835–843

A CONTRIBUTION OF ANCIENT CHINESE ALGEBRA: SIMULTANEOUS EQUATIONS AND COUNTING RODS

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We shall discuss some educational possibilities connected to the use of an ancient Chinese artifact. Our theoretical framework is based upon classical works by Vygotskij, Wartofsky and some recent papers by Bartolini Bussi. We proposed to a group of 10-11 year-old pupils a problem (from the Chinese treatise entitled Jiuzhang Suanshu, 1st cent. BC) that can be solved by a couple of simultaneous linear equations. Empirical data suggest that, in particular in the context of the game, the use of the primary artifacts (counting rods, the counting board), taking into account the secondary artifacts referred to the modes of action (rules and prescriptions), can be important in order to approach some mathematical contents.

INTRODUCTION

The main focus of this paper is on the educational possibilities connected to the use of an ancient Chinese artifact. A recent research (Bagni, forthcoming) considers an experience based upon the counting-rods and the counting board. In this paper we shall make reference to a procedure described in the chapter 8 (*Fangcheng*) of the *Jiuzhang Suanshu* (*Nine Chapters on the Mathematical Art*, 1st cent. BC), an anonymous handbook consisting of 246 problems. Clearly this method (i.e. Gauss elimination) can be performed with any kind of numerals. The focus or our work is not just calculating with Chinese rods or learning *fangcheng*: in fact, we shall analyse pupils' behaviour in order to point out the role of the considered artifacts.

The idea to use ancient Chinese methods for solving a system of equations can be considered as a suggestion for doing pre-algebra. Let us notice that in a positional number system a group of rods represents a set whose size is determined by its position on the table; in a linear equation, a group of rods represents an unknown set whose size, as we shall see, is determined by the box in which it is arranged. Thus, the Chinese notion of a system of linear equations is basically formed by their way of representing numbers.

The traditional Chinese representation of numbers by counting-rods on a board can be referred to the fingers of a hand. The counting-rods are arranged in columns placed side by side, with the rightmost column representing the units, the next column representing the tens, and so on. Before the 7th-8th cent. AD, there are no known written Chinese symbols which may be interpreted as zero (Martzloff, 1997), so in order to avoid misunderstandings Chinese mathematicians used two different dispositions: the aforementioned *Tsung* disposition for units, thousands and so on, and the *Heng* for tens, hundreds and so on (Figure 1; let us remember that from 200 BC Chinese mathematicians were used to representing positive and negative numbers by red and black counting-rods).

(Tsung) 1 2 3 4 5 6 7 8 9 (Heng) 10 20 30 40 50 60 70 80 90

Figure 1: Tsung and Heng counting-rods dispositions

Our goal is mainly of a cognitive nature: to study how pupils reproduce the rules of number manipulation and, in particular, to study the relationship between artifacts and cognition in a particular context (as we shall see, in a context related to the game). Our aim is therefore to discuss a *case study* whose importance is based upon the following elements: the subject is not included in the usual cultural system, so the influence of previous activities is rather small; moreover, it is relevant to the connections between "seeing", "doing" and "saying". We shall explain this point in the following section.

THEORETICAL FRAMEWORK

In a Vygotskijan perspective, the function of semiotic mediation can be connected to technical and psychological tools. Wartofsky identifies technical gadgets as *primary artifacts*; *secondary artifacts* are used in order to preserve and to transmit the acquired skills or "modes of action" (Wartofsky, 1985). So counting-rods can be considered as primary artifacts; prescriptions and representative rules (expressed in original books and commentaries, e.g. in the *Nine Chapters on the Mathematical Art*) are secondary artifacts. A mathematical theory is a *tertiary artifact* which organizes the secondary artifacts and hence the models constructed in order to represent the modes of action by which primary artifacts are used (Bartolini Bussi, Mariotti, & Ferri, 2005).

We must consider the distinction between artifact and *tool* (Rabardel, 1995), i.e. the artifact associated to a personal or social *schema* of action. If we make reference to an object as artifact, in order to be able to consider it as a tool, we need a constructive mediated activity on the part of the subject (Radford, 2002), so the considered artifact must be framed into a wide social and cultural context. When a pupil uses a primary artifact with reference to a secondary artifact, he or she *follows the rules*, so uses the primary artifact in a *rational* way. There is an important socio-cultural element in this point, taking into account that rule-following must be framed in an essentially collective practice.

An aspect to be considered is the context in which we are going to propose an activity with counting-rods: as a matter of fact, pupils will not be asked to approach an explicit *mathematical* (in particular: algebraic) activity. Let us remember the following remark by Lakoff and Núñez (2000, p. 30):

> "Mathematical abilities are not independent of the cognitive apparatus used outside mathematics. Rather, it appears that the cognitive structure of advanced mathematics makes use of the kind of conceptual apparatus that is the stuff of ordinary everyday thought".

With particular regard to algebraic procedures, Steinbring underlines that in order to record algebraic relations, one does not necessarily require the typical algebraic signs (Steinbring, 2006). So we are going to investigate whether the (non-typically algebraic) context of the game can be useful in order to allow the internalisation of some mathematical contents.

According to Bruner's constructivist approach, we create our own realities through interaction with the social world and with symbols (Bruner, 1987): learning itself must be considered within a cultural context, which involves the shared symbols of a community, its artifacts, and its traditions. A key concept is that of *context embeddedness*, where the term refers to the institutional and cultural context (Godino & Batanero, 1997). As we shall see, the method they used can be approached as a *new game*, which is not included in the usual cultural system, not as a traditional mathematical task.

METHODOLOGY

Let us now summarize the *fangcheng* method. In order to solve a system of linear equations, ancient Chinese mathematicians placed coefficients and numbers on the counting board (concerning the intercultural aspect, it is interesting to remember that ancient Chinese mathematicians took out lines as columns, and this from right to left). Their arrangement can be changed according to the following rules:

(1) the expression *biancheng* ("multiplication throughout", Martzloff, 1997, p. 253) is an instruction to multiply all the terms of a row by a given number;

(2) the expression *zhichu* ("direct reduction") carries out a series of term-by-term subtractions of a row from another row.

We are going to summarize an experience regarding a group of 11 year-old pupils. The experience took place in a 6th grade classroom (in Treviso, Italy): pupils were in the very first period of the Italian middle school; they came from several different primary schools. They did not know equations (only some of them, in the primary school, had dealt with simple exercises like "what is the unknown sum, if you know that three times half of this sum is…") and they did not know negative numbers. The experience took place during a lesson in the classroom; all the pupils (i.e. 18 pupils), the teacher and the researcher were present (the researcher was not the teacher of the pupils in question).

Let us summarize the three stages of our experience: firstly, the teacher presented to the pupils the expression of numbers by counting-rods. He proposed several examples and the pupils were invited to represent some numbers. Secondly, a counting board with labels was created and simple problems were represented by counting-rods. Finally, the pupils were divided into six groups of three pupils. The teacher proposed the following problem (*problem 1*) and represented it by counting-rods on a counting board; then the teacher underlined that "the rods arrangement represents problem data" and presented (with several examples, too) rules (1) and (2) by which rods arrangements can be changed:

(*Problem 1*) Suppose we have 5 bundles of type A cereals and 3 bundles of type B cereal, amounting to 19 *dou* of grain. Suppose we also have 3 bundles of type A cereals and 2 bundles of type B cereals amounting to 12 *dou* of grain. Question: how many *dou* of grain in 1 bundle of type A and type B cereals respectively?

This problem (based upon an original Chinese problem, with some variations of data) leads to the couple of simultaneous equations, in our modern notation:

$$5x + 3y = 19$$
$$3x + 2y = 12$$

Let us see (Figure 2) the rods arrangements proposed by some pupils in order to obtain the solution (we report the original counting board used; the translation of the labels is: "type A" and "type B bundles", "cereals"). In particular, a group of three pupils obtained the correct solution of the problem: one of the pupils (S.) proposed the steps and another pupil (F.) made some interesting comments; the third pupil's (R.) role was not active. During the experience, the teacher gave no suggestions; he just pointed out the mistakes.

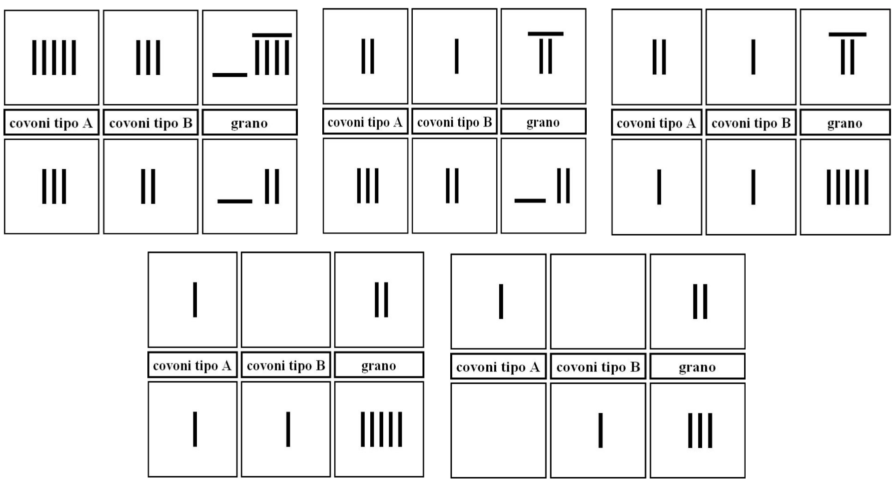

Figure 2: Rods arrangements in order to obtain the solution of the first problem

In order to deepen our comprehension of pupils' behaviour, we proposed to the same group of pupils another problem:

(*Problem 2*) Suppose we have 4 bundles of type A cereals and 1 bundle of type B cereal, amounting to 6 *dou* of grain. Suppose we also have 2 bundles of type A cereals and 3 bundles of type B cereals amounting to 8 *dou* of grain. Question: how many *dou* of grain are there in 1 bundle of type A and type B cereals respectively?

This problem leads to the couple of simultaneous equations (in our modern notation):

$$4x + y = 6$$
$$2x + 3y = 8$$

Let us see (Figure 3) the rods arrangements proposed by some pupils in order to obtain the solution. Of course it is impossible for the pupils to solve *problem 2* by using only rule (2): its application would imply the use of negative numbers.

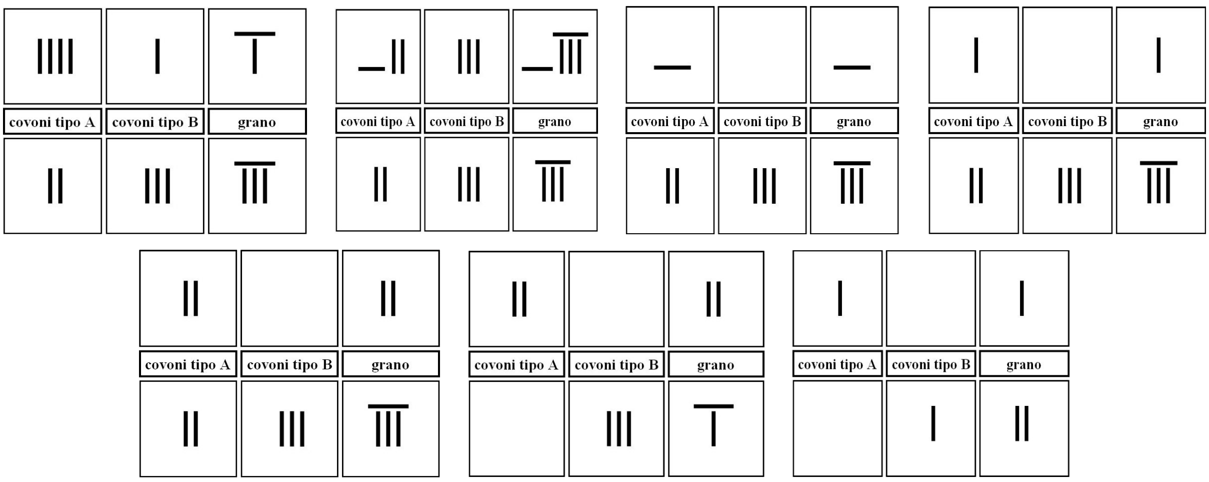

Figure 3: Rods arrangements in order to obtain the solution of the second problem

DATA

Recorded audio material and transcriptions allowed us to point out some salient short passages that were analyzed using techniques of qualitative research (see: Bagni, forthcoming). The experimental excerpts allowed us to highlight some elements that are relevant to our discussion.

Let us now examine the following transcript of the resolution of *problem 2*. It is particularly representative because, as we shall see, it gives us the opportunity to highlight some elements related to pupils' use of the artifacts (other data and results are not very interesting); the following transcript is to be considered as a *case study*:

[1]	Pupil S.:	Mm, no, we cannot take off these from those, there are not enough of them there. [S. indicates the second row, then the first row and touches the rods of the first square; she looks at the teacher]
[2]	Pupil F.:	The other also is impossible… Well, this time the exercise is impossible, isn't it? [S. touches the rods of the second row several times. One minute goes by]
[3]	Pupil S.:	Alright, we must take off the rods, but if we now increase them… using the other rule, we can multiply these, that is… yes… until there aren't enough of them.
[4]	Pupil F.:	Or those in the upper row.
[5]	Pupil S.:	Let's do it here … by three. [S. indicates the second square of the first row]
[6]	Pupil F.:	Oh yes, yes, we must make this equal to these! [She indicates the second squares of the first row and of the second row]
[7]	Pupil S.:	Go on, here, let's do it here by three. This one becomes three, four times three is twelve and the grain, six, gives eighteen. [She changes the arrangement of the rods]
[8]	Pupil S.:	Now we take them off… that is, here they goes away, twelve, ten, and ten here, too. [S. takes off the rods]
[9]	Pupil F.:	They are equal.
[10]	Pupil S.:	The first is done. [S. takes off the rods]
[11]	Pupil S.:	Now I need to get rid of that. [She indicates the first square of the second row] We just have to multiply by two. [S. adds the rods; then S. takes off the rods]
[12]	Pupil F.:	Done.
[13]	Pupil S.:	Well, we can divide in the usual way, six, we have two. [S. takes off the rods] The grain is one here and two here.

Let us now discuss pupils' behaviour. The first stage (steps 1-2, with a brief period of impasse, one minute without utterances) is interesting: there is a conflict between the situation and the necessity of taking off as many rods as possible (as pupils did in the previous resolution: see Figure 2). This initial conflict led F. to a wrong conclusion. But the utterance [3] by S. is very important:

[3] "Alright, we must take off the rods, but if we now increase them… using the other rule, we can multiply these, that is… yes… until there are enough of them".

These last words are meaningful: it is necessary that the counting-rods are "enough". The teleological structure of the action has a primary role. When pupils correctly used the rule (1) allowing the multiplication of all the terms of the second row by $k = 3$, F. realized that her previous conclusion ("this time the exercise is impossible") was wrong, she indicated the different squares and noticed:

[6] "Oh yes, yes, we must make this equal to these".

It is worth noting that the pupil connected the application of the considered rule to the counting-rods arrangement and to its effect upon this arrangement: F. was finally aware of the necessity of the absence of rods in a square in order to solve the problem, and she realized that in order to reach this situation it is necessary that two squares referring to the same type of bundles contain the same number of counting-rods. So solving strategies are connected to a combination of enactive skills (spatial awareness) and iconic skills (visual recognition and ability to compare: Bruner, 1966).

DISCUSSION

In the experience considered, pupils approached an algebraic procedure without using the typical algebraic signs. They effectively solved a couple of simultaneous linear equations by using counting-rods, so with reference to the secondary artifact expressed in chapter 8 of the *Jiuzhang Suanshu*. By that we do not suggest to introduce systems of linear equations in grade 6 (in many countries, e.g. in Italy, this is done in grade 8, or in grade 9); nevertheless the idea to use the ancient Chinese methods for solving linear equations for introducing the topic in school teaching can be interesting.

According to Bruner (1987), developmental growth considers the enactive, iconic and symbolic modes, and requires ability to translate between them: the experience considered provides us with interesting examples of translation (see moreover the examples discussed in: Bagni, 2006). An effective translation from the enactive to iconic mode (the frequent use of deictic expressions and of gestures, e.g. in steps 1, 5 and 6) and, in addition, a first approach to the symbolic mode can be seen in the pupils' behaviour.

Pupils gave a preference to the rule that is based more directly upon the concrete presence of counting-rods on the counting board (in the first resolution, see Figure 2). Indeed, when they apply rule *zhichu* ("direct reduction"), which allows term-by-term subtractions of two rows, they consider two quantities that they can see and touch; whereas when they apply rule *biancheng* ("multiplication throughout"), which allows the multiplication of all the terms of a row by a number *k*, this number *k* cannot be referred to the concrete presence of counting-rods. So, according to the empirical data, we can state that using (original) primary artifacts with reference to (original) secondary artifacts can be relevant to the introduction of some methods; more generally, the crucial point is that *the considered method is based upon the "positional" character of ancient Chinese algebra*, according to which a particular place in the board must be always occupied by a particular kind of number, e.g. a particular coefficient. This "positional" character cannot be pointed out in our basic algebraic European tradition.

Let us point out an important element: the considered artifacts are "non transparent" (in L. Meira's words: Meira, 1998). In particular, this means that there is no way of knowing, for instance, why some entries in the given rectangular array should be added together, or why the entries in the last column would represent totals, or why the given rules would preserve relevant features of the situation represented by the rods arrangement. Nevertheless, pupils effectively used a representation including signs, spatial relations and embodied rules with reference to a context having some typical features of a *game*. So an important path to follow can be related to the role of the game: this concrete context may allow a first construction of meanings that can be referred to abstract algebraic representation.

It is worth noting that the secondary artifact introduced is not strictly necessary in order to allow a physical action with the primary artifact. From this point of view, the introduced rules can be considered conventional, arbitrary (the original secondary artifact can be simplified: counting-rods can be arranged in very many ways; of course the intercultural aspect leads us to make reference to the original dispositions). So pupils made reference to a particular algebraic "language" that is not just a code, whose power can be referred to its syntax; its creative power lies in how the language itself is embedded into the rest of pupils' activities (Steinbring, 2006), and, in the case considered, into a *game*, a *new game* to be explored and played (we are dealing with a game making reference to a very different cultural tradition): and this can be *useful in order to give sense to the algebraic procedure considered*.

Of course it is important to investigate the conceptualization of the experience: further research will be devoted to the study of the educational possibilities connected to activities similar to the one considered (although, in our opinion, the main point cannot be summarised in the possibility of a complete derivation of an argumentation or a mathematical proof).

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