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A paradox of Probability: an experimental educational research in Italian High School

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Abstract. An informal point of view can be important and interesting in order to introduce the concept of Probability. In this paper we describe an experimental research activity about a first approach to Probability: we presented to students aged 16-17 years a short test based upon a well known paradox. The greater part of the pupils considered by intuition Laplace definition and applied it, but sometimes they made errors and this is caused by affective elements, too.

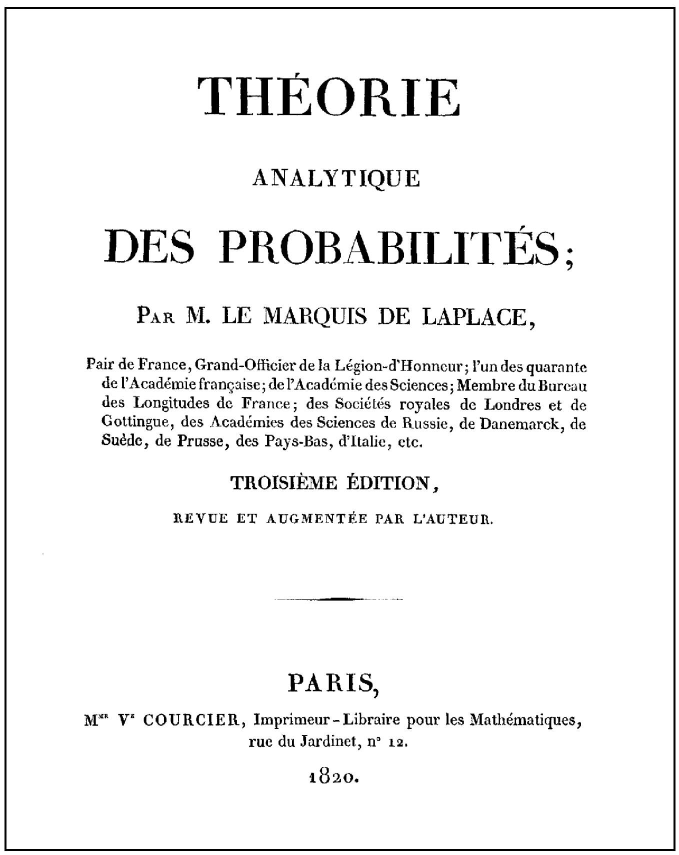
INTRODUCTION

In this paper we propose the beginning of a research activity on a first approach to Probability from an informal point of view: we shall consider some experimental results in order to point out reactions and obstacles and to evaluate them.

We shall not give a full presentation of researches upon the didactic introduction of Probability (Gagatsis, Anastasiadou & Bora-Senta, 1998, with an interesting summary; as regards the historical point of view, see: Daston, 1980; Lakoma, 1998; Todhunter, 1965; Maistrov, 1974). We underline that, according to E. Fischbein (1975 and 1984; Fischbein, Nello & Marino, 1991), teaching of Probability would begin with reference to pupils aged 12-14 years; but sometimes, for instance in Italian School, pupils' approach to main concepts of Probability takes place only in High School (pupils aged 16-17 years).

In particular, we wanted to investigate if classical Laplace ideas about the introduction of concept itself of Probability (the original work is: Laplace, 1820) are present in an intuitive approach to the matter. We made reference to *General Principles of Probability* stated by Pierre Simon de Laplace (1749-1827) in his famous *Essay philosophique sur les probabilités* (1814):

"1st Principle. It is the definition of Probability itself, which [...] is the ratio of the number of favourable cases and the number of all possible cases. 2nd Principle. It needs that all different cases are equally possible. If they are not so, it needs to find the respective possibilities, and this is one of the most difficult points of all the Theory" (Laplace, 1820).



THÉORIE

ANALYTIQUE

DES PROBABILITÉS;

PAR M. LE MARQUIS DE LAPLACE,

Pair de France, Grand-Officier de la Légion-d'Honneur; l'un des quarante de l'Académie française; de l'Académie des Sciences; Membre du Bureau des Longitudes de France; des Sociétés royales de Londres et de Gottingue, des Académies des Sciences de Russie, de Danemarck, de Suède, de Prusse, des Pays-Bas, d'Italie, etc.

TROISIÈME ÉDITION,

REVUE ET AUGMENTÉE PAR L'AUTEUR.

PARIS,

Mme Ve COURCIER, Imprimeur-Libraire pour les Mathématiques, rue du Jardinet, n° 12.

1820.

Third edition (1820) of P.S. de Laplace's *Theorie Analytique des Probabilités*

We proposed a test in which evaluation of probabilities can be easily related to Laplace ideas; however, according to A. Sfard, "there is probably much more to mathematics than just the rules of logic. It seems that to put out finger on the source of its ostensibly surprising difficulty, we must ask ourselves the most basic epistemological questions regarding the *nature of mathematical knowledge*" (Sfard, 1991, p. 2). Then we wanted to investigate if Laplace introduction is always (or frequently) intuitively adopted by pupils or if it is not so.

As we shall see, our test is based upon a paradox of Probability Theory; let us remember some words by G.J. Székely: "Just like any other branch of science, mathematics also describes the contrasts of the world we live in. It is natural therefore that the history of mathematics has revealed many interesting paradoxes some of which have served as starting-points for great changes" (Székely, 1986, p. XI).

METHOD OF OUR RESEARCH

We considered 52 High School pupils that did not know Probability from a formal point of view (3rd class of Italian *Liceo Scientifico*, pupils aged 16-17 years, in Treviso, Italy). We proposed to them the following test (see: Lolli, 1998, pp. 106-107; the paradox is quoted in: Székely, 1986, pp. 135-136, and in: Pflug, 1981, too):

In a room, there are two boxes, a white one and a black one, both containing liquorice and peppermint candies. A young boy, named Pierino, likes liquorice candies and does *not* like peppermint candies. In particular, there are:

Room 1

White box	Liquorice candies: 50	Peppermint candies: 60
Black box	Liquorice candies: 30	Peppermint candies: 40

Question 1. Pierino wants to get a candy from one box. Do you think that it's better for him to get it from the white box or from the black box?

Let us consider moreover two different boxes, in a different room, once again a white one and a black one, containing:

Room 2

White box	Liquorice candies: 60	Peppermint candies: 30
Black box	Liquorice candies: 90	Peppermint candies: 50

Question 2. Pierino wants to get a candy from one box. Do you think that it's better for him to get it from the white box or from the black box?

Room 3

Now both white boxes are poured in a new big white box and both black boxes are poured in a new big black box.

Question 3. Pierino wants to get a candy from one of these new big boxes. Do you think that it's better for him to get it from the big white box or from the big black box?

Of course, correct answers to both questions 1 and 2 are: "white box". In fact it is easy to calculate:

Room 1.

probability to get a liquorice candy from *white* box: $\frac{50}{110} = 0{,}45\ldots$

probability to get a liquorice candy from *black* box: $\frac{30}{70} = 0{,}42\ldots$

Room 2.

probability to get a liquorice candy from *white* box: $\frac{60}{90} = 0{,}66\ldots$

probability to get a liquorice candy from *black* box: $\frac{90}{140} = 0{,}64\ldots$

As regards the boxes in room 3, let us notice that the total numbers of the candies are:

Room 3.

White box	Liquorice candies: 110	Peppermint candies: 90
Black box	Liquorice candies: 120	Peppermint candies: 90

So the probabilities are:

Room 3.

probability to get a liquorice candy from *white* box: $\frac{110}{200} = 0{,}55\ldots$

probability to get a liquorice candy from *black* box: $\frac{120}{210} = 0{,}57\ldots$

Well, of course, as regards room 3, Pierino's best choice is to get his candy from the new *black* box! This correct answer can be clearly deduced by Laplace definition of Probability, but it is possible that some pupils try to give this or other answers without adopting it. So we wanted to examine if some pupils adopt Laplace point of view as regards the first part of the test (room 1 and room 2) but *not* as regards the last one (room 3).

The results of the test (as previously noticed, with reference to 52 High School pupils) are given in the following tables:

Answer to question 1	*Pupils*	*Percentage*
It's better to get the ca ndy from *white* box	38	73 %
It's better to get the candy from *black* box	10	19 %
No answer	4	8 %

Answer to question 2	*Pupils*	*Percentage*
It's better to get the candy from *white* box	43	82 %
It's better to get the candy from *black* box	5	10 %
No answer	4	8 %

Answer to question 3	*Pupils*	*Percentage*
It's better to get the candy from *white* box	33	63 %
It's better to get the candy from *black* box	12	23 %
No answer	7	14 %

Total time allowed: 10 minutes.

After the test, pupils were asked to justify briefly their answers. The greater part of the pupils (63%) that preferred the answer "white box" to question 3 underlined that answers to questions 1 and 2 ("white box"), according to Laplace definition of Probability, made it…immediate to give the s ame answer to third question *without* applying once again such definition. So the analogy and the links between questions 1-2 and question 3 caused several mistakes.

CONCLUSIONS

We do not think that the obstacles previously examined can be considered as properly epistemological ones or (only) as educational ones (see for example the fundamental classification in: Brousseau, 1983; Vergnaud, 1989). If we consider them as educational obstacles, we must underline that the influence of affective aspect is surely remarkable. Then, in our opinion, they can be regarded as *affective obstacles*, too, and it is difficult to overcome them completely just by educational means (like for example showing of counterexamples; see interesting situations described in: Kaldrimidou, 1987).

Situations previously described show that "simple" situations (often seen as natural and reassuring) are sometimes extended to a lot of cases, without deep and particular controls: this behaviour can cause inconsistencies and dangerous mistakes (as regard inconsistencies, see for instance: Tirosh, 1990).

Of course, we underline that analogical reasoning should not be too quickly dismissed: in fact, many mathematicians used and use it as one of the main ways for creating new mathematics! However, the really different propensity for self-correction should be considered, when we compare research mathematicians and young students: for example, frequently mathematicians employ analogical reasoning in formulation of a conjecture, whose logical soundness must be deeply verified; on the other hand, generally students do not perform this meta-discursive monitoring.

In our opinion important and interesting situations connected with the use of analogical reasoning can be particularly analysed by further researches.

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