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FUNCTIONS: PROCESSES, PROPERTIES, OBJECTS

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ABSTRACT

In this paper, some aspects of the leaning of real functions are investigated, with particular reference to secondary school pupils (14-19 years old). As regard action views, object-oriented and property-oriented approaches, several works indicate that the role of representations is fundamental. We propose a case study and conclude that, in order to make it possible the reification, it is important to consider and analyze the teacher's role in the institutionalization of the concept.

FUNCTIONS AND THEIR REPRESENTATIONS: A THEORETICAL PREFACE

The study of different representations of mathematical objects is a crucial issue of the didactics of mathematics (many references can be mentioned; for instance: Gagatsis & Christou, 2002; D'Amore, 2003a; Mousoulides & Gagatsis, 2004). We shall examine some representations of functions in order to compare different approaches (in particular we shall consider the *property-oriented* approach: Kieren, 1990) and to analyze the teacher's role in the institualization of this concept.

Concerning the links between human experience and formal mathematical systems, in Piagetian tradition, the distinction between mental structures and physical structures is frequently compared with the distinction between the signified, internal, and the signifier, external; only one of those levels can be observed, and the interaction between them is cyclical: mental activity can take place independently by physical activity, but mental structures themselves can be considered as the product of physical actions (Kaput, 1993).

The distinction between internal (mental) representations and external representations is often introduced by several researchers, explicitly or implicitly. But a consideration of these kinds of representations in terms of opposition can cause problems: first of all the cognitivist notion of mental representation is not quite clear; moreover, many important mathematical representations hardly can be framed into such distinction (Kaput, 1999, quotes for instance the Cartesian plane). In this paper we shall consider some representations of the functions, without asking ourselves if they are to be considered external or internal, because we do not consider too strict and rigid classifications sustainable and advantageous in this field.

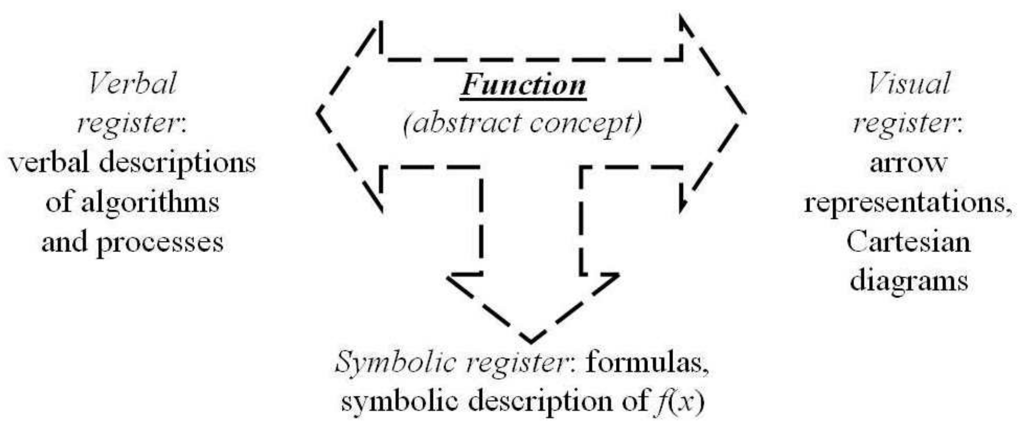
The use of traditional systems of formal representation implies educational restrictions: for instance, the consideration of the mathematical relations that can be algebraically represented can keep away the formal mathematics to the real experience. So some Authors suggest a wider use of new methods, based upon technology (Kaput, 1991 and 1993). This situation implies the *legitimation* of the new formal representations (Radford, 2002a, p. 236), with two linked aspects: political and epistemological (Radford moreover underlines that it is not possible to deal with the problem of the representations of knowledge without considering the ontological dimension: Radford, 2002a, p. 237). Moreover it is necessary to study the connections between spatial and temporal experience (for instance, body movements: Lakoff & Núñez, 2000), whose importance was underlined by several researches, and the activity of symbolization (Radford, 2002b and 2003a; it is generally interesting a revision of some ideas by Vygotsky concerning relations between representations and cognition, taking into account symbolic aspects as constitutive elements of the culture; see for instance L. Radford's *Cultural Semiotic Systems of Signification*: Radford, 2003b). It is worth noting that according to Habermas (1999) the rationality itself has three roots, strictly related the one to the others: the predicative structure of knowledge at institutional level, the teleological structure of the action and the communicative structure of the discourse (see the remarks with reference to argumentation and proof in: Balacheff, 2004; some ideas related to Set Theory are discussed in Bagni, forthcoming-c).

Important educational researches show that learning of the function concept is often facilitated by the early consideration of an action or of its interpretation as a process, where correspondences between quantities (numbers, magnitudes etc.) really take place: «We have a facility for thinking about processes or sequences of actions that can often be used to good effects in mathematical reasoning. One way to think of a function is as an action, a process that takes the domain to the range» (Thurston, 1994, p. 165; moreover: Briedenbach, Dubinsky, Hawks & Nichols, 1992). Let us quote A. Sfard's opinion that we shall remember frequently in our work, according to whom the development of «abstract mathematical objects» can be considered as the final product of the comprehension of processes (Sfard, 1991; Sfard & Thompson, 1994).

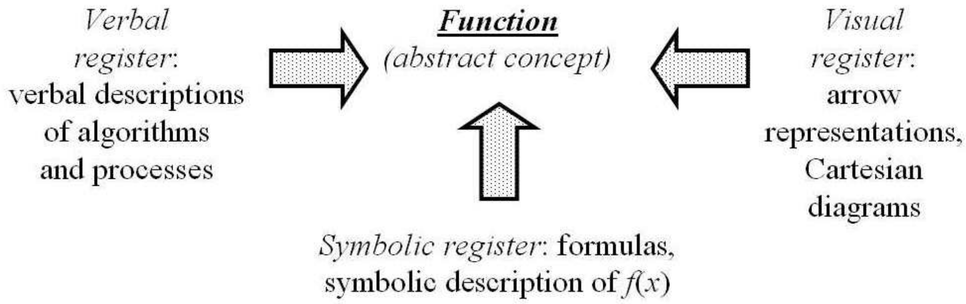
In this theoretical framework, semiotic aspects are important (we find suggestions in order to use different representations in the introduction of function concept since the late Eighties: Kaput, 1989). S. Vinner (1992, p. 197) underlines that a visual representation can be translated into verbal form, but this translation is not the first form that we remember when we think to the concept: in fact, in order to consider an action, a process and, finally, a mathematical object, many representations are needed.

When the *concept image* is formed, and this step allows us to reach the *concept definition* (Tall & Vinner, 1981, p. 152), representations' role is a primary one (according to S. Vinner, learning of mathematical objects needs examples and counterexamples stimulating the formation of the *concept image*: Davis & Vinner, 1986. The *concept image* and the *concept definition* are introduced in: Vinner & Herhkowitz, 1980; see moreover: Tall, 1988 and 1991; Vinner, 1983 and 1991); and the educational use of technology, too, is very important, frequently in order to support visual representations (Briedenbach, Dubinsky, Hawks & Nichols, 1992).

Of course, from the ontological point of view, the conception of mathematics that can be called Platonic, in which mathematical objects exist independent of their representations, cannot be stated uncritically. However, in educational practice, the concept of function is clearly approached by representations, mainly devoted to situations where the function is a process which explicitly connects two quantities.



Previous figure would be correctly considered: in fact, from the educational point of view, and in particular with reference to learning processes, arrows would not be oriented from the "abstract mathematical object" to its representations, but in the opposite way.



According to R. Thom, «as regards the interaction *Meaning-Object*, clearly the meaning (…) generates the object (…). And the object generates the meaning, too, when we interpret the representation» (Thom, 1974, p. 233): surely this process is *twofold*: of course if we should limit the mathematics education to the generation and the consideration of (single) representations, we should wrongly grant a strong privilege to some particular aspects of the general concept, to the detriment of others.

Relevant issues concern the connection between the acquisition of a representation (by that we are making reference to semiotic) with the full and effective conceptual acquisition of a mathematical object (noetic), in particular with reference to the object "function" (D'Amore, 2001b; see moreover: Duval, 1998, D'Amore, 2001a and 2003b; the diverse hypotheses that lie at the foundations of unsuccessful devolution – Perrin Glorian, 1994– are presented and discussed in: D'Amore, 2003a).

FROM PROCESS TO OBJECT

If the learning of the function concept is just devoted to the consideration of an action or of a process (and of their representations) it would be incomplete and ineffective: mathematical concepts' formation is, as previously noticed, a complex process, it needs a sequence of steps, so a progressive approach (however, as we shall see, several researchers have recently suggested models that are not strictly sequential: Slavit, 1997, p. 268). A. Sfard calls *reification* the passage from the consideration of a process to a conception that properly considers the mathematical object: and we should have a conception *object-oriented* (D. Slavit however notices any «lack of clarity» when an *object-oriented* comprehension of a mathematical idea is stated: Slavit, 1997, p. 265, Thompson, 1994); she underlines that if we make reference to «mathematical *objects*», we must be able to consider the *outputs* of some processes without considering processes themselves (Sfard, 1991, pp. 9-10). If we accept that this concept formation takes place by a *hierarchy*, according to which a step cannot be reached before all previous steps are completed (Sfard, 1991, p. 21; Sfard & Linchevsky, 1994), we must conclude that the full development of these steps must be carefully controlled by teachers (Fischbein, 1993) and that in this development several representations are needed. An important problem is the correct and effective realization of this reification: if we try to force a structural point of view, we cause the formation, in pupils' minds, of dangerous pseudo-objects and misconceptions.

According to M. Artigue (1998), who summarizes the «pioneer work» by E. Dubinsky (1991; Sfard, 1992), a hierarchy can be conceived from the early consideration of an action to the following conception of a process (*interiorization*), so of a mathematical object (*encapsulation*). The notion of *procept* (Gray & Tall, 1994) is very interesting: Artigue (who quotes particularly: Tall, 1996) points out that the notion of procept concerns and underlines symbols' roles. Really several mathematical symbols can be considered as procepts: «sometimes they represent a process and sometimes the result of the process» (Artigue, 1998). Once again semiotic representations' role is relevant.

Theoretical frameworks previously sketched recently evolved and abandoned a strictly hierarchic and sequential approach: connections between processes and objects remarkably granted importance to dialectic dimensions of various steps and to semiotic aspects of conceptualization activities.

FUNCTIONS AND PROPERTIES

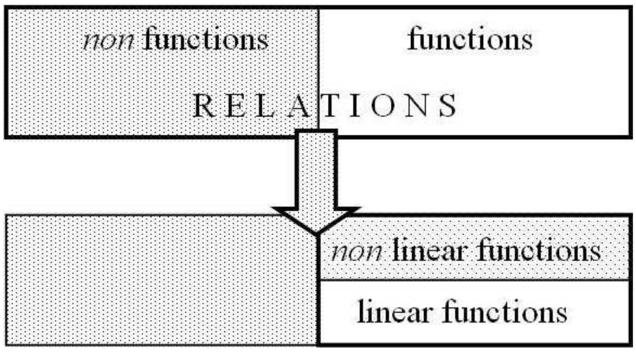
So semiotic aspects are very important in order to define an effective theoretical framework for the learning of mathematical concepts, and in particular of the function concept (D'Amore, 2001a and 2001b). This makes it necessary to consider carefully some connected needs: for instance, in order to obtain a full learning, it is not enough to have a development of single registers, but a coordination of such registers is needed (Duval, 1995b, p. 259; see the interesting paper devoted to multiple intelligences: Fredens, 2004). Moreover, the great importance of representations (for instance visual or graphic representations) in the *property-oriented* approach to function concept must be remembered (Kieren, 1990): this approach does not replace previously quoted theories, but just proposes a new interpretation of them (Slavit, 1997, p. 269).

According to this approach, a function can be introduced and described with reference to its local properties (intersections, maximum and minimum points, vertical asymptotes etc.) and to its global properties (symmetry, periodicity, invertibility etc.): clearly a *property-oriented* approach, whose educational role is quite important, deals with pupils' ability to establish connections between different representations (Nemirowsky & Rubin, 1992; Monk & Nemirowsky, 1994), frequently with reference to graphic technology (Ruthven, 1990). Different features, and in particular the different flexibility, of visual and symbolic representative registers can lead to a different generality so to a different possibility of such registers to be employed: this fact can constitute an obstacle with reference to the correct learning of the function concept, particularly if the coordination of representative registers is lacking. In the following paragraph we shall consider this important question by a *case study*.

Educational experience itself allows us to state that the study of local and global properties of a function (we mean, particularly, properties of its Cartesian graph) is fundamental in order to characterize important classes of functions: for instance, linear functions' Cartesian graphs are characterized by some evident global properties and this allows us to identify a linear function as a function whose graph has got the considered global properties; and similarly as regard other important classes of functions, e.g. quadratic functions, periodic functions etc.; on the contrary, main properties that characterize continuous functions (or derivable functions) are local.

Some experimental researches (quoted in: Slavit, 1997, p. 272) pointed out that frequently pupils use either approaches based upon the consideration of a real correspondence (*action view*, *operational view*) or approaches based upon the consideration of some particular properties (*property-oriented*). For instance, when they study a function (we now make reference particularly to Italian High School, 18-19 years old pupils), they sometimes consider either correspondences between some values of variable x and their $f(x)$, or global and local properties of the considered function. Form the semiotic point of view, the early approach is referred to numeric representations, the latter one to graphic representations, so to the visual register.

A *property-oriented* approach is useful in order to classify different functions, and moreover in order to identify common features of some functions; the general the function concept itself, as a particularization of the general concept of relation, can be introduced by a *property-oriented* approach: for instance, a function $D \rightarrow \mathbf{R}$ is a relation (so a subset of $D \times \mathbf{R}$ or of $\mathbf{R} \times \mathbf{R}$, being $D \subseteq \mathbf{R}$: however by that we do not state that this introduction must be considered educationally effective!) whose Cartesian graph meets *just once* a line parallel to y axis, i.e. whose equation is $x = a$ (being $a \in \mathbf{R}$; or it meets in *exactly one point* a line whose equation is $x = b$ being $b \in D$).



If we summarize previous example, we can notice that a *property-oriented* approach is generally based upon a partition of a set I in a subset S constituted by elements characterized by the property P and in the complementary subset S' constituted by elements that are *not* characterized by the property P. So clearly an important step is the indication of the *environment*, so of the set I that will be divided into subsets S, S' as previously described. For instance, with reference to the general function concept, before introducing the relations that are characterized by the quoted property, it is necessary to fix what we mean by "relation".

GENERALIZATION VERSUS PARTICULARIZATION?

Previous considerations about *property-oriented* approaches can lead to learning processes based upon *particularization*: from the (general) concept of relation we consider the (general) function concept, then a particular class of functions (e.g. linear functions) and so on. However, everyday educational experience suggests us an opposite path: a particular correspondence leads to its *generalization* in a whole class of functions in order to consider, finally, the general concept.

early consideration
of a concrete correspondence (e.g. a linear function)

⇓

first generalization
the class of linear functions (to whom the considered correspondence and other ones belong)

⇓

further generalization
"any" functions (e.g. linear and non-linear functions)

Every passage from a class (a set) of functions to another wider class leads us to consider new correspondences (concerning problems in comparing infinite sets: Tsamir & Tirosh, 1994 and 1999):

- having any *common* features with correspondences previously considered (for instance with reference to their Cartesian graphs);
- having any *different* features from correspondences previously considered (for instance with reference to their Cartesian graphs).

A path towards the function concept can be, for instance, the following.

- We can consider the single correspondence that associates any number to its double:

$$\boxed{x \to 2x}$$

(in the Cartesian plane: a *single line* whose equation is: $y = 2x$)

- Then we can consider:

$$\boxed{x \to mx}$$

(or: $x \to mx+q$, their graphs are *straight lines*)

- A further generalization leads to:

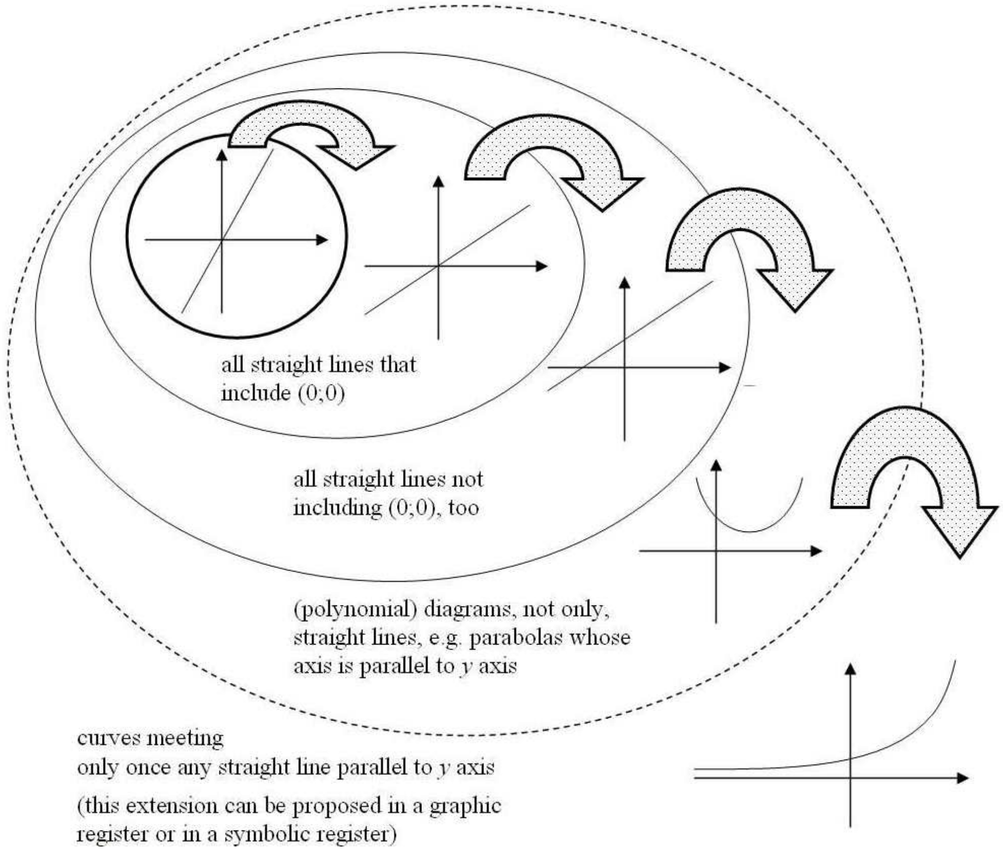
$$\boxed{x \to f(x)}$$

(their Cartesian graphs are not straight lines, but they meet *just once* a line parallel to y axis)

Of course the described path is just a first example: in the following table the detailed process described on the right includes the process described on the left:

Considered correspondences	*graphic representations*	*considered correspondences*	*graphic representations*
the (particular) corespondence $x \to 2x$	line whose equation is $y = 2x$	the corrispondence $x \to 2x$	line whose equation is $y = 2x$
↓	↓	↓	↓
		linear functions (properly considered)	lines whose equations are $y = mx$
		↓	↓
functions that can be represented by a line	lines whose equations are $y = mx+q$	functions that can be represented by a line	lines whose equations are $y = mx+q$
↓	↓	↓	↓
		polynomial functions	examples of lines, parabolas etc.
		↓	↓
“any” functions	curves meeting only once any straight line parallel to y axis	“any” functions	curves meeting only once any straight line parallel to y axis

Let us underline once again the fundamental role of counterexamples: pupils must realize that the proposed generalization is a real one.



It is not always easy to distinguish the classes by their graphic representations: for instance, we can hardly distinguish between "polynomial functions" and "general functions" (in the previous figure the boundary line is... light). As regard graphics of polynomial functions, we can suggest the absence of asymptotes, but such property cannot be referred only to polynomial functions (e.g. let us consider the Cartesian graphic of $y = x+\cos x$: no asymptotes!). So sometimes a symbolic register can be more effective than a graphic one.

Concerning *property-oriented* approaches, another interesting element must be underlined: when we deal with the general function concept (and similarly when we deal with particular classes of functions) we often consider the visual representation, in particular the Cartesian graph. This fact can cause obstacles (Bagni, 1997b): the main importance granted to the visual representation can lead to forget the importance of elements that would constitute parts of the function definition itself (for instance, the explicit indication of the domain: Bagni, 1997a) and this causes remarkable difficulties. By that we conclude that teachers must carefully control, in the learning and in the everyday practice, the independence of mathematical object's and representations' roles.

So *property-oriented* approaches, whose educational importance is clearly a primary one, do not solve completely the problem of the reification, i.e. of the final building of mathematical objects. D. Slavit notices that there are no works that prove whether a *property-oriented* approach effectively improves the full development of an *object-oriented* conception of function (Slavit, 1997, p. 271).

AN EXPERIMENTAL CASE STUDY

The consideration of particular properties (mainly referred to graphic representations of a function) is surely useful, as above underlined, in order to characterize the examined function or a whole class of functions, but it can, sometimes, cause doubts and obstacles, too. In order to clarify the sense of the latter provocative statement we are going to propose an experimental case study (Gagatsis, Michaelidou & Shiakalli, 2002; Evangelidou, Spyrou, Elia, & Gagatsis, 2004; Gagatsis & Shiakalli, 2004).

Let us report an educational experience in which a teacher (*Tea.*) and a pupil (*Anna*) are involved. Anna is an university student, (1st year, Biologic Sciences); she is 19 years old and previously studied in a *Liceo scientifico* (Italian High School); at the moment of the reported experience she was studying Calculus and Biostatistics, including elements of Calculus (limits, derivatives, integrals; no differential equations were treated), Geometry and Statistics; the described experience took place during an individual meeting with the teacher. The teacher gave the pupil the following exercise:

Complete: what does $y = f(x)$ represent in the Cartesian plane?

$f''(x) = x$ $\qquad\qquad$ $f(x) = \ldots\ldots\ldots$

Anna: Well, the second derivative is *x*. I must integrate by the usual formula *(she writes)*:

$f''(x) = x$ $\qquad\qquad$ $f(x) = \ldots\ldots\ldots$

$$\int x^n = \frac{x^{n+1}}{n+1}$$

Anna (after any seconds): Isn't it correct?

Tea.: Check it carefully.

Anna: Oh, yes, I have to put *dx* and *c (she writes)*.

$f''(x) = x$	$f(x) = \ldots\ldots\ldots$
$\int x^n dx = \frac{x^{n+1}}{n+1} + c$	

Anna: Now I integrate the given function.

$f''(x) = x$	$f(x) = \ldots\ldots\ldots$
$\int x^n dx = \frac{x^{n+1}}{n+1} + c$	$\frac{x^2}{2}$
	$\frac{1}{2}\frac{x^3}{3} = \frac{x^3}{6}$

Anna: So $f(x)$ is $\frac{x^3}{6}$, I can complete my exercise.

$f''(x) = x$	$f(x) = \frac{x^3}{6}$
$\int x^n dx = \frac{x^{n+1}}{n+1} + c$	$\frac{x^2}{2}$
	$\frac{1}{2}\frac{x^3}{3} = \frac{x^3}{6}$

Tea.: It is not completely wrong, but what about integration constants?

Anna (immediately): Oh, excuse me, I always forget it *(she writes +c in the result and in the passages)*.

$f''(x) = x$	$f(x) = \frac{x^3}{6} + c$
$\int x^n dx = \frac{x^{n+1}}{n+1} + c$	$\frac{x^2}{2} + c$
	$\frac{1}{2}\frac{x^3}{3} = \frac{x^3}{6} + c$

Tea.: Is it correct, now?

Anna: Yes, so it seems to me. I don't see what to add, now.

Tea.: I repeat what I've said before: it is not wrong, but perhaps it is not the general solution of your exercise. Why did you write the integration constant?

Anna: I must put it, when I integrate. If I derive, all constants vanish.

Tea.: Well. And now, how many integrations have you done?

Anna: Two.

Tea.: So don't you need two constants?

Anna: Perhaps it's so: however, c or $2c$, it's the same thing.

The presence of two successive integrations is unusual for Anna, who frequently integrated a sum of functions: in this case we can point out an *Einstellung* effect; some clauses of the *didactical contract* probably induced the pupil to use rules that would be correct only in different situations.

Tea.: Explain to me the reason.

Anna: When I integrate, I put $+c$ and when I derive, it vanishes. Similarly $+2c$, too, would vanish. I am quite sure: if I derive twice my result then I obtain x.

Tea.: Sure, it's true: if you would derive twice, you would get back to x. However try to solve once more your exercise, so integrate x twice, separately, a first time and a second time. (*Anna takes a new sheet of paper and writes).*

$$f''(x) = x \qquad f'(x) = \frac{x^2}{2} + c \qquad f(x) = \frac{1}{2}\frac{x^3}{3} + c + c = \frac{x^3}{6} + 2c$$

Tea.: Look at the last passage, from f' to f: did you integrate… everything?

Anna: Oh, must I integrate c, too?

Tea.: What do you think?

Anna (after any seconds): Now I think so: I never realize it before (*she writes*).

$$f''(x) = x \qquad f'(x) = \frac{x^2}{2} + c \qquad f(x) = \frac{x^3}{6} + cx + c$$

Tea.: Just a moment: must your two constants be equal?

Anna: I don't think so.

Tea.: In this case it's better to use different letters, for instance c, k (*Anna writes*).

$$f''(x) = x \qquad f'(x) = \frac{x^2}{2} + c \qquad f(x) = \frac{x^3}{6} + cx + k$$

Tea.: Now it's alright. Go on.

Anna: I must draw it in the plane; but I have some letters, so I cannot do it. *(Pause).* I can give any values to the letters; as usual, when we are dealing with integrals.

Tea.: What values do you give to c and to k?

Anna: All real numbers: I can choose any real number.

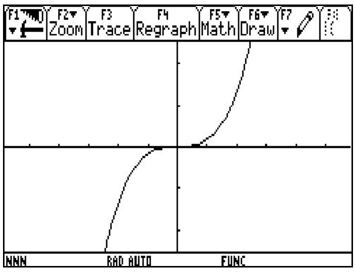
Let us underline that the employed register is only the symbolic one: the visual register was not considered yet. Now the exercise forces the pupil to trace the graphic representation of the considered function, so the register must be changed.

Tea.: For instance I suggest you the three cases (1) $c = 0$, $k = 0$; (2) $c = 0$, $k = 1$; (3) $c = -1$, $k = 0$. Now, by your graphic calculator, you can trace corresponding diagrams. Draw all of them on a sheet and compare them: finally, tell me what is your opinion *(The teacher writes values and formulas)*.

(1) $c = 0$ and $k = 0$;	(2) $c = 0$ and $k = 1$	(3) $c = -1$ and $k = 0$
(1) $y = \dfrac{x^3}{6}$	(2) $y = \dfrac{x^3}{6}+1$	(3) $y = \dfrac{x^3}{6}-x$

(Anna traces diagrams by her graphic calculator and draws them).

(1)



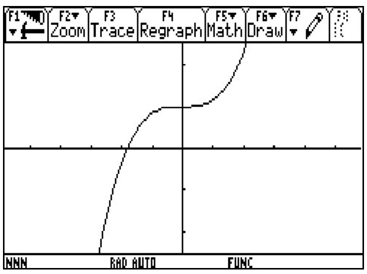
Anna (after any seconds): I see that the first and the second are equal, while the third graphic is different.

Tea.: In my opinion the first is not equal to the second.

Anna: I mean that those curves are just the same curve differently placed. The third curve's shape is different.

Tea.: And in your opinion what does it mean?

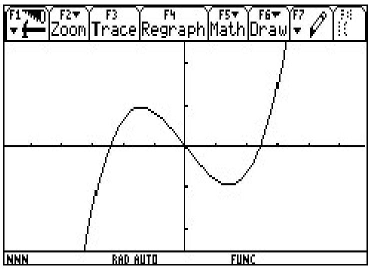
(2)



Anna: Probably there is something wrong. If a curve is right, if it is a solution, of course the other curve is not right! I can accept that the same curve can be differently placed, as it happens in the cases (1) and (2), alright: when we work with integrals it frequently happens. But in this case it's the curve itself that changes! (*Pause*). The last curve meets in three different points the x axis, other curves meet it just once.

(3)

Tea.: But you have obtained your three curves working on the same problem and you didn't make any mistakes: your graphic calculator is a good friend! So



all your curves would be considered acceptable.

Anna (after any seconds): If I should have to choose a curve I should prefer (3): it seems to be general, there is the term with x, too. In (1) and (2) c is 0, there is not the term with x, so they are particular cases.

So Anna uses either the visual register or the symbolic one and she considers some properties of examined functions (D. Slavit notices that the presence of the graphic representation induces pupils to consider function's properties: Slavit, 1997, p, 273); in fact she realizes that the presence of three intersections of the curve with x axis is caused by the presence of $c \neq 0$.

(In order to interpret these answers according to the theory of the *intuitive rules* by D. Tirosh see for instance: Stavy & Tirosh, 1996). But she does not accept yet a "solution" constituted by a family of functions whose graphic representations are not congruent curves ("the same curve differently placed", in Anna's words).

Tea.: Don't you think that all of them can be solutions of your exercise? Remember that all of them were obtained from the equation $y = \frac{x^3}{6}+cx+k$. Many other curves, if we choose different values of c and k, are solutions.

Anna (after any seconds): So the graph does not exist…

Tea.: Well, a single graph doesn't exist because a single function doesn't exist: there is infinity of curves, a curve for every couple of values k and c.

Anna: But they are different curves!

Tea.: Sure: don't you think it's possible?

Anna (after a pause): An exercise would have only one solution: only one diagram.

Let us notice that this statement appears to be inconsistent with some considerations previously stated by the pupil herself (a lot of references are devoted to problems connected to inconsistencies; we just quote: Tall, 1990; Tirosh, 1990); however we shall see that they are different references to kinds of representations that, in this case, are not mutually coordinated.

Tea.: Several exercises have got more than a single solution: few minutes ago you gave the solution of the exercise by the equation $y = \frac{x^3}{6}+cx+k$. For different, particular values of c and k this is a particular equation, so it represents a particular function.

Anna: That is different: $y = \frac{x^3}{6}+cx+k$ is *one* equation. I can write it, it is *one* thing *(she emphasizes "one")*, although it can be transformed in many different things by changing c and k. But how can I write or draw many graphs, all together? When I consider curves I must choose one of them. Usually, dealing with integrals, all curves are equal, they are just differently placed, as (1) and (2): but (3) is a different curve, this is the problem. And many other curves, too, will be different.

So Anna reaches the crucial point: she explicitly states that *in the symbolic register* it is possible to express by *one* writing a class of function, but this *cannot be done in the visual register*, when we consider Cartesian graphs (the pupil herself observed: «I must draw it in the plane; but I have some letters, so I cannot do it»).

So the different "flexibility" of employed representative registers, symbolic and visual, embodies the difference of (potential) generality, so different possibilities of utilization of registers themselves: and such discrepancy can constitute a remarkable obstacle, with reference to the learning of the function concept and of the connected procedures (that, as we shall underline in the final paragraph, can be considered as a relevant part of the concept), particularly when the coordination of different representations is lacking (Duval, 1995a). In order to overcome such difficulties, it is necessary a final explanation by the teacher, whose role is to achieve a clarification and an institutionalization of the roles of representations and of the properties that can be gathered.

FINAL REFLECTIONS: FROM PROCESS TO OBJECT INSTITUTIONALIZATION AND TEACHER'S ROLE

We shall fix some considerations regarding the very important step that leads to the reification and the role of the teacher, whose relevance was pointed out in the previous paragraph; let us underline once again that following conclusions would not be referred to a strictly sequential development of the examined processes: in fact, as previously noticed, the learning of a mathematical concept can be broken down in many steps, but in our opinion such breakdown is not to be considered rigorously, for instance from the chronological point of view, even though some steps must be considered preparatory, propaedeutic to other; however we prefer to present a progressive dialectic comparison of steps, with new reflections and deepening of previous steps, too.

The passage from the early consideration of an action to the conception of a process (interiorization) is referred only to the considered case, so to a single, particular situation: so pupils deal with an example of the mathematical object that, in the future, will be generally considered as *function*. A *property-oriented* approach allows the specification of particular features of a class constituted by such examples, particularly if the game is now played with main reference to representations (mostly visual representations). And the reification itself can be considered with reference to a whole class of functions (e.g. in the case of linear functions, previously remembered, linearity can be conceived as the arrangement of some properties: Slavit, 1997, p. 275).

A remarkable difficulty in the building of the abstract object, so of the true mathematical object, is the generalization (Eisenberg & Dreyfus, 1994); in fact the importance of properties employed in order to characterize some classes of functions must be reduced to the correct size: a function can have such properties, and in this case we shall say that it is a function of a particular kind, for instance a linear one, but it can have *not* them, and in this case, too, it is a function (sometimes there is no symmetry between "having a property" and "having not it": e.g. it is easy to point out that a continuous function is frequently considered by several pupils as the standard case, while a function that is not continuous is considered as a very particular one: Bagni, 1997b; overgeneralizations, in the case of linear functions, are examined in: Markovitz, Eylon & Bruckheimer, 1986; Bagni, 2000). A mathematical object, we can say a *relation*, if we have previously introduced this concept, must have different properties in order to be considered, generally, a *function*. However a *property-oriented* introduction of the functions among relations would be, as noticed, strictly linked to representations, in particular to the consideration of properties of the Cartesian graph.

We previously stated that teachers must work in order to lead pupils to a full clarification of the roles of used representations and of the mathematical object. Really the teacher plays a primary role in the step that leads from the consideration of a process to the building of a mathematical object, so in the *reification* (Brousseau, 1986; Perrin-Glorian, 1997): the teacher, in a didactic situation, must carefully verify that various elements that are going to constitute the *concept image*, and, later, a full *concept definition*, keep their correct roles; then he proposes to pupils the final generalization, by relevant examples and counterexamples, by classifications and references to semiotic representations (that in the present step are not predominant).

So the reference to disciplinary epistemic analysis is fundamental. Of course in this work we do not want to propose a disciplinary status of mathematics: however educational activities need some indications, so a social agreement regarding main features of mathematical activities. Pupils must comprehend some main features of mathematics, and such learning cannot forget, for instance, social aspects. According to J.-Ph. Drouhard, we can say that the teacher must give pupils the «3rd order knowledge» that, beyond technical aspects (definitions and theorems, so «1st order knowledge») and deduction's and representation's rules («2nd order knowledge», for instance frequently used, as previously noticed, in the examination of several functions' properties), will allow pupils themselves to reach the awareness of the level of their own learning (Drouhard, forthcoming; moreover: Robert & Robinet, 1996). In our opinion, the awareness of these features of learned mathematical knowledge is a fundamental element of all teaching-learning process related to mathematics and will complete the knowledge that will be employed by pupils in order to engage a mathematical activity. Further research will be devoted to the study of this final step.

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