Revista Latinoamericana de Investigación en Matemática Educativa. Número Especial. Semiótica, Cultura y Pensamiento Matemático (2006). Radford, L. & D'Amore, B. (Eds.), 247–266

EVERYDAY AND MATHEMATICAL LANGUAGE
100 YEARS AFTER THE PUBLICATION OF "ON DENOTING" BY BERTRAND RUSSELL

Giorgio T. Bagni
Department on Mathematics and Computer Science
University of Udine (Italy)

SUMMARY. The article "On denoting" by B. Russell, published in 1905, is a milestone in philosophical reflection on language. In the present paper we examine pupils' reactions both to a sentence inspired by a celebrated example and to a statement expressed in mathematical language. Experimental data can be interpreted with reference to the concepts of truth and rationality and as a consequence we propose some reflections that shift "the standard of epistemic objectivity from the private certainty of an experiencing subject to the public practice of justification within a communicative community" (J. Habermas). We conclude that the language is a very important moment in which the meaning of an expression is fixed, but we must keep in mind that "language, like any other semiotic system, functions inside a cultural network of significations" (L. Radford).

KEY WORDS: language, justification, meaning, rationality, truth and validity.

1. INTRODUCTION

Many recent works show that culture and mathematical thinking are strictly linked, both in the history of mathematics and in mathematical education (a number of references can be quoted; see for instance: Wartofsky, 1979; Crombie, 1995; Radford, 1997; Furinghetti & Radford, 2002). Of course the language is a very important element of the culture: it is well-known that Aristotle himself distinguished men from animals because of the presence of the λόγος (*logos*, often translated by "reason"; but H.G. Gadamer suggests a proper translation of this term by "language": Gadamer, 2005, p. 155). It follows that the careful study of language has to be considered as a crucial point of the research in mathematics education.

A quotation by L. Radford (making reference to Ilyenkov, 1977, p. 79) will help us to frame more precisely the focus of our work and its educational relevance: Radford states that "language is *one* of the means of objectification (albeit a very important one), but that there are also several others"; moreover, "as a means of objectification, language does not objectify indiscriminately. Language, like any other semiotic system, functions inside a cultural network of significations, from whence grammar and syntax draw their meaning" (Radford, 2003a, p. 141; see moreover: Radford, 2003b). So in this paper we are going to ask ourselves: firstly, can we consider the language as a sort of favourite or absolute moment in which the meaning of an expression is fixed? (Let us notice, for instance, that paradigmatic analysis seeks to identify the different pre-existing sets of signifiers which can be related to the content of texts: Sonesson, 1998). Secondly, let us remember that, according to R. Rorty, the discipline presently called philosophy of language has two different sources: one of them is the cluster of problems "about how to systematize our notions of meaning and reference in such a way as to take advantage of quantificational logic"; the latter, explicitly epistemological, "is the attempt to retain Kant's picture of philosophy as providing a permanent ahistorical framework for inquiry in the form of a theory of knowledge" (Rorty, 1979, p. 518). In this paper we are going to discuss, on the basis of some experimental data, this approach: as a matter of fact, can we always make reference to a quite sure set of meanings for linguistic expressions and, in particular, to a clear notion of truth?

From the historical viewpoint, G. Vattimo points out that "almost all the most important and subtle problems of contemporary language philosophy were elaborated and faced, for the first time, in the Middle Ages" (Vattimo, 1993, p. 640; in this paper the translations are ours). The medieval doctrine of *suppositio* is deemed significant (Bocheński, 1956, pp. 219-230; Kneale & Kneale, 1962). According William of Shyreswood, "meaning is the representation [*praesentatio*] of an idea in the mind. The *suppositio* is the co-ordination [*ordinatio*] of the concept under another concept" (Bocheński, 1956, p. 217); Petrus Hispanus, too, in his *Summulae logicales*, pointed out the difference between *significatio* and *suppositio* (Geymonat, 1970, I, p. 549; Bagni, 1997); and in his *Summa Logicae* (I, 63, 2) William of Ockham (1281-1349) stated that the *suppositio* "is a property belonging to a term, just because [it is included] in a proposition" (Bocheński, 1956, p. 219).

Nevertheless we cannot develop completely this interesting issue through reference to the Logic of the Middle Ages; we shall introduce the subject of our study through a theoretical framework based upon some elements of 20th-century philosophical research: in section 2 we shall make reference to the paper *On denoting* by Bertrand Russell (1872-1970), published a century ago,

together with its historical connection to Meinong and Frege (2.1); some positions of Wittgenstein's (2.2), Quine's and Brandom's (2.3) will allow us to introduce Apel's and Habermas' approaches (2.4), which are to be considered crucial for our work. Through these we shall discuss (section 5) experimental data (sections 3 and 4).

2. THEORETICAL FRAMEWORK

2-1. Frege and Russell

Let us consider first some reflections on "definite descriptions" (Penco, 2004, p. 54): we shall compare some ideas put forward by Gottlob Frege (1848-1925) and by Russell. In order to introduce the problem, it is recalled that since Aristotle we have known that "through language we can correctly refer to things that do not exist [τὰ μὴ 'όντα] or to elements whose existence is possible but that can hardly be proved" (Lo Piparo, 2003, p. 165). It is moreover worth mentioning the theoretical approach of Alexius Meinong's (1853-1920), who stated that "objects of knowledge do not necessarily exist" (Meinong, 1904, p. 27; Orilia, 2002).

The Fregean approach is based upon the Compositionality Principle (Frege, 1923, p. 36), according to which a statement containing a term without denotation has no truth value: for instance, a statement referring to a non-existing person is neither true nor false (Frege, 1892). On the contrary, according to Russell, statements containing definite descriptions (e.g. *the current President of Italian Republic*) implies the existence of the individual (*Mr. Carlo Azeglio Ciampi*) to whom the considered property is referred (and this individual is unique), at least at the time when the sentence is stated (March 2006). The problem is that some definite descriptions (and names are definite descriptions too) do not refer to existing individuals: when we talk about *Ares* or *the father of Phobos and Deimos* we do not make reference to an existing individual.

In order to avoid ambiguity, in his article entitled *On denoting*, published in *Mind* a century ago, Russell suggested making the logical form of a definite description explicit. So a proposition like *The father of Phobos and Deimos is the Greek god of war* would be *There is one and only one individual of whom it can be said: he is the father of Phobos and Deimos, and he is the Greek god of the war*. Frege's and Russell's approach are very different: let us consider, for instance, the sentence *The King of France is bald*: according to Frege it is neither true nor false because the term *the king of France* has no reference; according to Russell it is false because we can write it in the

form: *There is one and only one king of France and he is bald* (Wittgenstein will make reference to a similar position: Wittgenstein, 1969a, p. 173).

Many years after the publication of *On denoting*, P.F. Strawson (1950) underlined an important distinction between a *sentence* and an *utterance* and this led us to distinguish between *denotation* and *reference*: as a matter of fact, denotation links an expression and what it denotes (taking into account conventions and linguistic rules); reference links an expression and the object to which the speaker wants to make reference (Bonomi, 1973; Penco, 2004, p. 84). In *The King of France is bald*, Russell deals only with denotation, while Frege considers the speaker's idea to make reference to a non-existing object, so he concludes that the sentence has no truth value, such a reference being impossible. Of course if we consider a different context, e.g. a legend or a fiction where the king of France is actually bald, we should have to revise our position (it should be remembered that according to Frege words must be considered only within a proposition: for instance, *Phobos and Deimos* could indicate either the sons of Ares and Aphrodite or the satellites of Mars; see: Frege, 1923).

2-2. Wittgenstein: from "Tractatus" to "Philosophical Investigations"

The position of Russell's most important pupil, Ludwig Wittgenstein (1889-1951), is rather complex because it must be divided into two very different periods. In his *Tractatus logico-philosophicus* (published in 1921 with a preface by Russell himself) Wittgenstein reprises (sometimes critically) and develops some ideas of Frege's and of Russell's: while Frege considers natural language as unavoidably imperfect, Russell wants to point out its logical form (Russell, 1910) and Wittgenstein states that "in fact, all the propositions of our everyday language, just as they stand, are in perfect logical order" (Wittgenstein, 1922, § 5.5563; but Wittgenstein's position expressed in his *Tractatus*, reveals some tension; see: Marconi, 2000a, p. 54); so if our language "looks ambiguous, we must recognise that its essence or its true logical form are hidden" (Penco, 2004, p. 60).

The so-called second Wittgenstein proposed a very different approach (his *Philosophical investigations* were published in 1953, two years after the philosopher's death): words' meanings must be identified in their uses within a context. The concept of «language-game» is fundamental: it is a context of actions and words in which an expression assumes its meaning; so a language game is both a tool for the study of the language and the "starting point: we can reflect on the language by describing the differences and similarities of language games, instead of looking for its essence, as

done in the *Tractatus*" (Penco, 2004, p. 105; concerning the continuity between the first and the second Wittgenstein, see: Marconi, 2000b, pp. 95-101). In addition, Hilary Putnam developed this approach and concluded that the meaning of a word is to be found in (and in some ways distributed among) the community of speakers (Putnam, 1992).

Let us now examine a remark by Habermas (that we shall reprise later): through his descriptive approach to the use of language, Wittgenstein levels its cognitive dimension; as soon as truth conditions that we must know in order to employ propositions correctly are derived just from linguistic praxis *to which we are used*, the difference between validity and social value vanishes (Habermas, 1999, p. 80): this suggests a revision of the concepts of «validity» and «truth». Of course Habermas' position must be considered critical: he underlines that the justification cannot be based upon life, but rather must be related to fundability (Habermas, 1983, p. 80). We shall reprise this point later.

2-3. Some ideas by Quine and Brandom

Willard Van Orman Quine (1908-2000) makes reference to the modality *de dicto* and *de re* (Quine, 1960; Kneale, 1962): "a *de re* belief is a belief expressed by the speaker about some properties of a certain object in the world; a *de dicto* belief is a belief expressed by the speaker about a proposition" (Penco, 2004, p. 161; interesting historical references can be found in: von Wright, 1951, pp. 25-28 and Prior, 1955, pp. 209-215). For instance, the proposition *John believes that Ares is the Greek god of war*, referring to a *de dicto* belief, cannot be replaced by *John believes that the father of Phobos and Deimos is the Greek god of the war*: as a matter of fact we cannot be sure that John knows that *Ares* is *the father of Phobos and Deimos*. On the contrary, the proposition *about Ares John believes he is the Greek god of the war*, referring to a *de re* belief, can be replaced by *concerning the father of Phobos and Deimos, John believes he is the Greek god of the war*, where the speaker characterised *Ares* through a personal description, even if John does not know it. Some similar situations have been studied by Frege (see for instance: Frege, 2001; Origgi, 2000, pp. 110-123) and we shall reprise them in order to discuss our experimental data.

Brandom tries to revise some of Wittgenstein's ideas and proposes replacing his language games with his «game of giving and asking for reasons» (Brandom, 1994 and 2000). Although Brandom's conception of language has been considered restrictive (it does not consider aspects like calling, ordering etc.), his approach will be relevant to our research (see moreover: Habermas, 1999, pp. 102 and 140).

2-4. Apel and Habermas

According to Karl-Otto Apel (1987), every speaker implicitly makes reference to norms for *meaningful* and intelligible discourse, *truth* (romantic correspondence between sentence and reality), *veracy* (correct expression of the speaker's state) and *normative correctness* (respect of community rules). As a consequence we are able to acquire the conditions for «ideal» communication, which assumes the role of normative principle: the discussion's impartiality and the possibility to reach some agreement among the bargaining parties depend on those conditions (see moreover the "rational" discussion as introduced in Lakoff & Johnson, 1980, p. 111, and the "conversation", p. 102).

According to Habermas, the rationality of judgements does not imply their truth, but only their justified acceptability in a particular context (Habermas, 1999, p. 102). Jürgen Habermas distinguishes between the truth of a statement and its rational affirmability (Habermas, 1999, p. 11) and reprises Apel's ideas (criticised in: Davidson, 1990) in order to highlight the fundamental possibility of an «ideal» communication: he underlines the importance of the inclusion in a universal world of well-ordered interpersonal relations, and the crucial element in order to do that is the rigorous condition of communication (Habermas, 1999, p. 279).

The intersubjective validity does not derive only from a convergence that can be observed with reference to the ideas of different individuals: Habermas refers epistemic authority to a community of practice and not only to individual experience (Habermas, 1999, pp. 136 and 238). The structure of the discourse creates a connection between the structures of rationality itself. As a matter of fact, it has three different roots, closely related the one to the others: the predicative structure of knowledge at an institutional level (Cassirer, 1958, III, p. 329), the teleological structure of the action and the communicative structure of the discourse (Habermas, 1999, p. 99). These Habermasian considerations will be very important in interpreting our experimental data.

3. METHODOLOGY

In this work we are going to analyse the behavior of a group of students aged 15-16 years (5^{th} class of a *Ginnasio-Liceo Classico*, in Treviso, Italy) dealing with a question about the truth of two sentences in some ways similar to *The King of France is bald* (Russell, 1905). The *Ginnasio-Liceo Classico* is a school with high educational standards, in which pupils are asked to study many classical subjects, in disciplines like Italian Literature, Latin ad Greek Literature, History and so on;

mathematical curriculum is based upon elementary Algebra and Euclidean Geometry, and some basic elements of Logic are included (in particular, pupils knew the notion of proposition as «statement that can assume one and only one truth value, true or false»: for instance, sentences including predicates related to subjective evaluations cannot be considered propositions).

During a lesson, in the classroom, pupils has been divided into groups of three pupils each. The division was at random. The researcher (who was not the mathematics teacher of the pupils but who was however present in the classroom with the teacher and the pupils) proposed two sentences to the pupils and invited all the groups to decide if such sentences were true of false. In particular, we are going to analyze the discussion developed into one of the groups: other discussions were not very "interesting", so we shall not consider them.

The question was proposed while taking into account the importance of avoiding the suggestion of a strict dilemma («true or false?») forcing the students to give a specific answer. As we shall see, the first sentence (*The King of the inhabitants of the Moon is bald*) makes reference to Russell's aforementioned example; after some minutes, the second sentence (*1/0+1/0+1 is odd*) was added, in order to take into account the influence of algebraic language. Of course a full evaluation of this important aspect ought to be based upon a particular and detailed research: in our opinion algebraic language's role can be very binding, so it would be considered not only with reference to the assignment of a truth value to an algebraic expression like the considered one.

4. Experimental data

The researcher writes the first sentence on the blackboard. The second sentence will be added after ten minutes:

For each sentence say: Is it a true sentence? Is it a false sentence?

(1) The King of the inhabitants of the Moon is bald (2) ...

Discuss your answer in the group and write it on a sheet of paper.

Here is the (translated) transcription of the conversation that took place in the group formed by A., B., C. (pupils' initials have been modified for deontological reasons):

4-1. Transcription

[01] A.: (*smiles*) "What is it?"

[02] B.: (*in a low voice*) "*The King of the inhabitants of the Moon is bald*."

[03] A.: "The King of the inhabitants of the Moon, what does it mean?"

[04] C.: “Well, I say, the Moon is something with no hair, if we consider the sun and its beams…”

[05] B.: (*ironically*) “But what are you talking about?”

[06] C.: “No, no, I am joking, there are no inhabitants on the Moon. If they existed I would be able to state something.”

[07] B.: (*looks around*) “But what does it mean, true or false?”

[08] A.: “I do not know who are the inhabitants of the Moon, and then, come on, there are no inhabitants on the Moon and so there is not a king.”

[09] B.: “Then it is false.”

[10] C.: “It’s not as easy as it seems, in my opinion there is something unclear. They are playing with words and so we don’t understand. Let’s read carefully. There is not a king, and the inhabitants, what does it mean? On the Moon there is nobody, hence the king of the Moon is the Moon itself.”

[11] A.: “Perhaps there are some micro-organisms, something that we cannot see, entities different from us.”

[12] C.: (*gesticulating*) “Or think, perhaps someone saw an astronaut with his helmet, so that he looks bald and when he talks about it, so perhaps it is true.”

[13] A.: (*sure*) “No, it is not relevant, it says the inhabitants of the Moon, it doesn’t say the Moon or the king of the Moon is false, I mean bald. We must see the inhabitants and then the king.”

[14] B.: “Well, in this case it’s false, there are no inhabitants, no king, hence of course he’s not bald.”

[15] A.: “Watch out, perhaps there’s a trap, as he says (*C.*), perhaps the exercise cannot be done.”

[16] B.: “I’ll divide this sentence up: when I say that the king of the inhabitants does not exist, full stop, it is false, and what follows is also false. If I say, later, that he is bald or not, this is not important, do you understand?”

[17] A.: (*doubtful*) “So let’s say that the sentence… would be false.”

[18] C.: “Yes, the simplest thing to do is to answer that it’s false. But if the question deals with a film or a tale with a king of the Moon that is bald, in that tale it’s true.”

[19] A.: “Just a moment, it’s better to emphasize the king of the Moon, in our answer. The king is false. If we want to say that the whole sentence is false we must be able to see the king, with his hair and …”

[20] B.: (*interrupting*) “No, it’s impossible to see him, he doesn’t exist. (*To C.*) It’s no tale, otherwise they would have told us. So it’s false.”

[21] A.: (*after a while*) “In short, one thing is to say that a sentence is false, I say that something is not true and so there is something wrong in the sentence. Another thing is to talk about someone and then say he is, for instance, bald or not, when I talk about a person I suppose he exists.”

[22] B.: “No, wait, but in your opinion is it enough to say something about someone who doesn’t exist in order to make him real? If he doesn’t exist, he’s false.”

[23] A.: “He is not false, the king, the problem is whether it’s false that he is bald. Let’s think carefully, before answering. It seems false, but perhaps it’s not so.”

[24] B.: “Listen, think about the question as a whole, they say the king is bald, it can be false because the king is not bald or because there is no king at all. If we want it to be true we must have the king and he must be bald.”

[25] C.: (*looking at A., a bit impatient*) “Come on, it’s clearly false! You make us wrong, if you say that it’s not false, then it is true, and what do you mean? Do you mean that the inhabitants of the Moon are bald?”

[26] A.: “Eh, it’s not true, it’s obvious. However it is not easy to understand. (*Looking at B.*) No, you are right, let’s write false, I agree.”

Now the Researcher completes the task on the blackboard:

For each sentence say: Is it a true sentence? Is it a false sentence?

(1) The King of the inhabitants of the Moon is bald (2) 1/0+1/0+1 is odd

Discuss your answer in the group and write it on a sheet of paper.

[27] B.: “Yes, it’s like before. False.”

[28] A.: (*doubtful*) “Just a moment… if we say false, it’s even. Maybe this exercise is impossible.”

[29] B.: “No, why do you think even? It’s different. Here it’s odd, we must look at this sentence.”

[30] A.: “Watch out, it’s not like the first sentence. And what about if they had said even?”

[31] B.: “False. It would be false, 1/0 is not a number.”

[32] C.: “1/0 means infinity.”

[33] B.: “No, the teacher told us it isn’t true, 1/0 is impossible.”

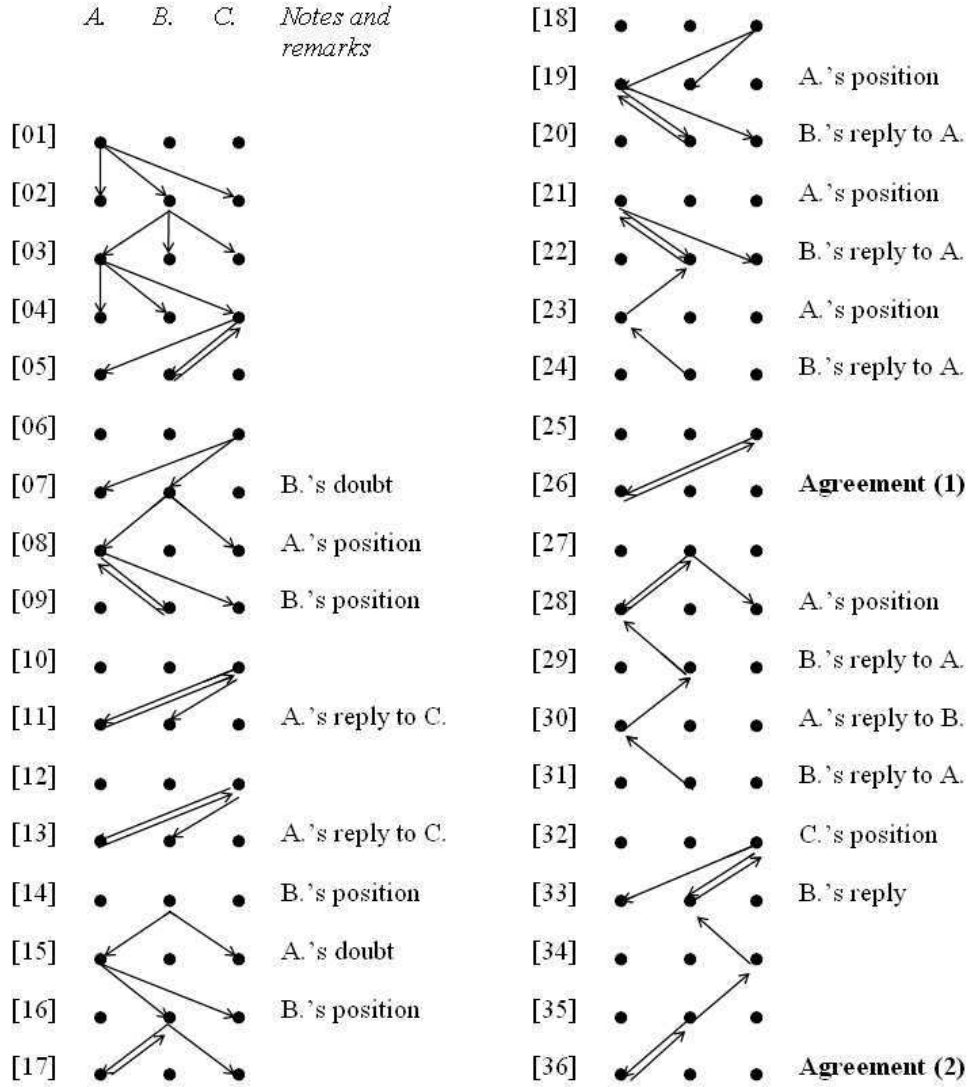
[34] C.: “It’s not infinity but it’s a very very big number. How can I say if it’s odd or even?”

[35] B.: "No, no, it's not a number, it would be very big but actually it doesn't exist."

[36] A.: "Come on, there is a trick: they make you think it's odd because it's like 2+2+1 that would be 5, but the starting number doesn't exist. It's false, once again."

4-2. Interaction flow chart

In the following flow chart (Sfard & Kieran, 2001; Ryve, 2004) different arrow directions are used to distinguish *proactive* and *reactive* utterances. In the case considered, the essential connection with everyday language prompted us to avoid the distinction between *object-level* and *non-object-level* utterances.



In the next section we are going to analyse our experimental data (transcriptions and flow chart) on the basis of our framework.

5. Discussion

5-1. First sentence

In [03] A. proposes the problem of reference and in [04] C. seems to suggest the possibility of an unusual interpretation of «bald» ("the Moon is something with no hair, if we consider the sun and its beams…"). However the student himself, turns back in [06] to a more usual meaning ("No, no, I am joking, there are no inhabitants on the Moon"). A.'s next utterance, [08], can be connected to the Compositionality Principle: "Come on, there are no inhabitants on the Moon and so there is not a king".

C.' utterance [10] is interesting: "they are playing words and so we don't understand. Let's read carefully. There is not a king, and the inhabitants, what does it mean? On the Moon there is nobody, hence the king of the Moon is the Moon itself". He doesn't recognise the "perfect logical order" of common language (Wittgenstein, 1922, § 5.5563): as well as «referential opacity» (Quine, 1960), he considers the semantic aspect and proposes an unusual *suppositio* (if "on the Moon there is nobody", we could say that "the king of the Moon is the Moon itself").

C.' next utterance [12] is also interesting ("Or think, perhaps someone saw an astronaut with his helmet, so that he looks bald and when he talks about it, so perhaps it is true"): the communication function of the language is explicitly considered (Dummett, 1993, p. 166; see moreover: Habermas, 1999, p. 105) and this is the one point in which falsehood, although in *de dicto* modality, does not refer only to the problem of existence. A's utterance [13] ("no, it is not relevant, it says the inhabitants of the Moon, it doesn't say the Moon or the king of the Moon is false, I mean bald. We must see the inhabitants and then the king") is not completely clear, but brings the discussion back to the main question.

Now we can consider the direct comparison of B.'s ideas with A.'s. In [14] B. says: "well, in this case it's false, there are no inhabitants, no king, hence of course he's not bald". A.'s utterance [15] expresses some doubts ("perhaps the exercise cannot be done"): he seems to choose a «Fregean» approach, and a conclusion avoiding the assignment of a truth value, but in [16] B. expresses his viewpoint further: "I'll divide this sentence up: when I say that the king of the inhabitants does not exist, full stop: it is false, and also what follows is false. If I say, later, that he is bald or not, this is not important, do you understand?" The Compositionality Principle is once again followed, but B.

seems to consider a «Russellean» denotation. A.'s utterance [17] ("so let's say that the sentence… would be false") does not show conviction.

C.'s utterance [18] refers to the importance of the context (see moreover the *suppositio*): now the connection between an expression's meaning and its use in a context is clear: "but if the question deals with a film or a tale with a king of the Moon that is bald, in that tale it's true".

In [19] A. declares his willingness to accept the falsehood of the sentence considered, but underlines that it mainly refers to the existence of the king of the Moon: "just a moment, it's better to emphasize the king of the Moon, in our answer. The king is false". This point is interesting: like in [17], A. shows a positive frame of mind with reference to B.'s position, but according to him "if we want to say that the whole sentence is false we must be able to see the king, with his hair and …".

After B.'s reply [20], taking into account C.'s objections too ("It's no tale, otherwise they would have told us") and after a while, in [21] A. says: "One thing is to say that a sentence is false. I say that something is not true and so there is something wrong in the sentence. Another thing is to talk about someone and then say he is, for instance, bald or not, when I talk about a person I suppose he exists." So A. seems to propose a distinction between a *de dicto* modality and a *de re* modality: the pupil would distinguish a statement like *I say that the king of the Moon is bald* and a statement like *I say about the king of the Moon that he is bald* (Penco, 2004, p. 191). The second expression, in A.'s opinion, would be divided up in the following way: *I am talking about the king of the Moon and (later) I say he is bald*: so the expressions examined would bind the speaker.

As we can see from the flow-chart, a direct comparison between A. and B. now resumes ([21]-[24]): B.'s reply [22] is interesting ("but in your opinion is it enough to say something about someone who doesn't exist in order to make him real?" This brings to mind Meinong's position according to which "objects of knowledge do not necessarily exist": Meinong, 1904, p. 27). Nevertheless A. is not completely persuaded and certainly, in this « game of giving and asking for reasons»: he acknowledges in [23] the plausibility of B.'s conclusions ("it seems false, but perhaps it's not so") but at the same time confirms his «Fregean» approach ("he is not false, the king, the problem is whether it's false that he is bald"). However, the first part of the discussion is about to finish: as a matter of fact, in [24] B. states once again his «Russellean» viewpoint: "listen, think about the question as a whole, they say the king is bald, it can be false because the king is not bald, or because there is no king at all. If we want it to be true we must have the king and he must be bald".

While [14], [16] and [22] did not completely persuade A., this utterance is crucial and conclusive (C.'s utterance [25], "come on, it's clearly false" can be compared with a well-known Wittgenstein's note: "all I should further say as a final argument against someone who did not want to go that way, would be: «Why, don't you see…!» – and that is no *argument*": Wittgenstein, 1956, I, § 34). In [26], after pointing out the lack of clarity in the expression examined ("Eh, it's not true, it's obvious, however it is not easy to understand": and A. makes reference to a «non-truth», perhaps in order to underline its difference from a «falsehood») A. accepts B.'s conclusions.

With reference to Apel's perspective, A.'s doubts do not seem to be related to comprehension of the meaning of the discourse: its «truth» (correspondence between sentence and reality) is connected with or perhaps set against its *normative correctness* (respect of community rules), mainly if we consider the features of a critical analysis of the sentence itself, of the "definite descriptions" (Penco, 2004, p. 54) that we find in it and of the coordination of its parts ([24]: "it can be false because the king is not bald, or because there is no king at all"). If we keep in mind the distinction between the truth of a statement and its rational affirmability (Habermas, 1999, p. 11) and if we interpret «correctness» as acceptability according to rigorous conditions of communication (Habermas, 1999, p. 279), we can say that A. is induced to accept the correctness of the shared final choice thanks to the argument developed by the group of students (in particular by B.). We shall reprise these considerations in the final section of our work.

5-2. Second sentence

B.'s role is now sure and, as shown by the flow-chart, the discussion about the second sentence can de divided into two moments: a first debate between A. and B. ([27]-[31]) and a second debate between C. and B. ([32]-[35]). In both these moments, B. expresses his positions properly, taking into account the results of the previous discussions about the first sentence (see for instance the utterance [27]).

A.'s doubt [28] is interesting (the utterance is similar to [15], but now it is based upon a different argument). According to A., to say that *'1/0+1/0+1 is odd' is false* would correspond to saying that *'1/0+1/0+1 is even' is true*: let us note that a similar argument (to say that *'The king of the inhabitants of the Moon is bald' is false* would correspond to saying that *'The king of the inhabitants of the Moon is hairy' is true*) was not considered by A. in the previous part of the discussion (only C.'s utterance [25] can be connected to this argument). Such a difference seems to

be related to the different contexts: the mathematical one, with its particular language and symbols, can suggest the use of *tertium non datur*.

B.'s strong utterance [31] ("1/0 isn't a number") is very important: the student interprets the sentence *1/0+1/0+1 is odd* as *1/0+1/0+1 is an odd number* and, more precisely, *1/0+1/0+1 is a number and this number is odd*. The first part of this sentence is false (the analogy with B.'s utterance [16] is clear: we have once again a «Russellean» denotation) so all the sentence must be considered false.

The discussion between C. and B. deals with the «nature» of 1/0: in [32] C. states "1/0 means infinity" and, because of B.'s objection ([33]: "no, the teacher told us it isn't true, 1/0 is impossible"), in [34] C. changes his mind and states that "it's a very very big number ", so "how can I say if it's odd or even?" However in [35] B. points out: "no, no, it's not a number, it would be very big but actually it doesn't exist" and the discussion leads A. to accept B.'s justified position explicitly ([36]: "the starting number doesn't exist. It's false, once again").

It should be noted that the syntactic structure $n+n+1$ to which the second sentence makes reference can lead the students to consider an odd number. This element is very relevant, and in our opinion this is the crucial point with reference to the role of algebraic language: in the first sentence, the existence of the king of the inhabitants of the Moon would have no consequences about his hair, but now if n is an integer, $n+n+1$ would really be an odd number (in [36] A. says that "they make you think it's odd because it's like 2+2+1 that would be 5, but the starting number doesn't exist"). But this factor did not influence the students.

6. CONCLUDING REMARKS

Let us now turn back to the questions proposed in the Introduction. Clearly experimental data can lead us to state once again that language is a very important moment in which the meaning of an expression is fixed; but clearly we must keep in mind that "language, like any other semiotic system, functions inside a cultural network of significations" (Radford, 2003a, p. 141). It is impossible to make reference to a completely sure set of meanings and to a single, absolute notion of truth (moreover, relevant issues concern the connection between the acquisition of a representation, namely a linguistic one, with the full conceptual acquisition of an object: D'Amore, 2001b; see moreover: Duval, 1998, D'Amore, 2001a, 2003a and 2003b).

The experience described brings to mind a position held by Putnam (1992) according to which the meaning (and we are thinking about a whole sentence, more than about a single word) is to be

found in the community of the speakers and refers to different ways of considering the sentence (and, as we shall see, to the three "different roots of rationality": Habermas, 1999, p. 99). Rorty notices that a merely «subjective» argument must be disregarded by the reasonable partners of a conversation (Rorty, 1979, p. 368): we realized that a meaning has been built by collective negotiation, a real «game of giving and asking for reasons» (Brandom, 2000); but in our opinion it is trivial to conclude that both arguments by B. and by A. are plausible (Strawson, 1950). As a matter of fact, this plausibility of both positions and their evolution lead us to answer: is it correct to propose a similar «truth evaluation»?

Of course both sentences were ambiguous, while the choice true-false can be considered only if the assigned sentence is a real «proposition»: but how can our pupils recognise real «propositions»? The traditional answer «a proposition is a statement that assumes one and only one truth value", in this case, can be circular. Moreover, it is important to realize that the ambiguity considered is not connected to the structure of the assigned sentences (for instance, *3/6+3/6+1 is odd* is clearly a… perfect proposition!).

The task considered is neither connected only to an isolated epistemic rationality, nor refers only to coherence (Rorty, 1979, p. 199; Williams, 1996, p. 267; certain and coherent proofs can coexist with "conceptual confusion": Wittgenstein, 1953, § II-XIV) or analogy: the comparison [27]-[31] demonstrates that the difference in the contexts (the first sentence is expressed in common language, the second refers to a mathematical context) does not authorize us to transfer the truth value from the first to the second sentence uncritically. Moreover, the term «false» can have different values in different contexts (Lakoff & Johnson, 1980 p. 153).

So should we doubt everything? This question is misleading ("if you tried to doubt everything you would not get as far as doubting anything. The game of doubting itself presupposes certainty": Wittgenstein, 1969b, § 115; from the logical viewpoint we agree with: Lolli, 2005, pp. 13-17). Furthermore, a charge of a conventionalistic reduction of the concept of truth would be groundless (Andronico, 2000, p. 252); Wittgenstein himself would reply: "«So you are saying that human agreement decides what is true and what is false?» – It is what human beings *say* that is true and false; and they agree in the *language* they use. That is not agreement in opinions but in form of life" (Wittgenstein, 1953, § 241).

As noted in 2-4, this position has been elaborated by some authors. It is important to consider our traditional notions of «truth» and «validity»: knowledge's objectivity criterion is founded on public

praxis instead of private certainty, so «truth» becomes a «three members» concept of validity (Habermas, 1999, p. 239), a validity justified with reference to a public (Schnädelbach, 1992).

The discussion of our experimental data does not allow us to conclude only that working together (in groups) is useful: such a conclusion would be induced by our opting to propose the exercise to some groups of pupils. The final common decision of the students was achieved after an active discussion, and has some consequences (Habermas, 1999, p. 137; in our case, for instance, the group must declare its decision to the Researcher, to the Teacher and to other students); so we must surpass the sphere of propositions (and texts) and take into account the sphere of actions, e.g. in using a predicate (as noticed by Kambartel, 1996, p. 249). With regard to students' behavior, the discussion (in the perspective of a decision to be taken) seems to interpret the mentioned position and to develop the different roots of rationality (Habermas, 1999, p. 99). Of course the debate, under the explicit influence of the text of the assigned exercise, is still far from the «ideal» communication described by Habermas and by Apel (C.'s role, for instance, is often minor, although his utterances related to the *suppositio* are really interesting); in other groups of students the discussion developed without a final agreement (Lakoff & Johnson, 1980); nevertheless our experimental data (in particular utterances [19], [21]-[24], [28]-[31] and [32]-[35], too) enables us to state that the discussion did not lead the pupils only to a convergence of different ideas, but to a real change of viewpoint (see Habermas, 1999, pp. 238 e 254). This fundamental moment can be highlighted in the utterances [24] and [35].

We would like to make a final reflection: we provided out students with a stimulating question about the truth (and the falsehood) of some sentences in different contexts, and this is a quite traditional exercise; but how can we speak about «truth» with any certainty? Rorty asks himself if the truth of a sentence can be really considered as independent from the context of the justification (Rorty, 1994) and our experience seems to bear out his doubt: the behavior of some students did change after the passage from a non-mathematical context to a mathematical one; for instance, in [28]-[30] and in [36] the influence of algebraic syntax is clear (A.: "they make you think it's odd because it's like 2+2+1 that would be 5, but the starting number doesn't exist"; let us remember that mathematical curriculum of the Italian *Ginnasio-Liceo Classico* includes several chapters devoted to algebraic syntax; nevertheless, as previously noticed, algebraic language's general role in pupils' behavior can be investigated more deeply).

Reflection on these issues is important (Lakoff & Johnson, 1980, pp. 197-222): a distinction between «validation» (*Geltung*) and «validity» (*Gültigkeit*) is fundamental and can lead us to

weaken the traditional distinction between the «validation» of a statement that is approved and the «validity» of a statement that *deserves* intersubjective acknowledgment because it is true (Habermas, 1999, p. 277). If we accept that a truth predicate can be considered (also) in the language game of the argumentation, we can point out its importance (also) with reference to its functions in this language game and hence in the pragmatic dimension of a particular use of the predicate (Habermas, 1999, p. 246) and we must take into account some important consequences. Truth itself must be related to a particular culture (to a particular language system): probably students belonging to different cultures would express their argument in a different way (as previously noticed, in Italy the *Ginnasio-Liceo Classico* is considered a school with high educational standards). Truth is relative to comprehension, so there are no points of view allowing us to obtain «absolutely objective truth» (Lakoff & Johnson, 1980, pp. 236 and 283).

So intercultural aspect must be considered, and this point is expressed in Wittgenstein, too: "if anyone believes that certain concepts are absolutely the correct ones, and that having different ones would mean not realizing something that we realize – then let him imagine certain very general facts of nature to be different from what we are used to, and the formation of concepts different from the usual ones will become intelligible to him" (Wittgenstein, 1953, § II-XII). This point of view has been examined by M. Messeri, who concludes: "so there is something intrinsecally misleading in ethnocentric behavior according to which different cultures are incomplete, rough and unsatisfactory" (Messeri, 2000, p. 190).

Moreover, some influences of the didactical contract can be considered: probably students' arguments would be different if used outside the school, in a different context. So does the predicate of truth have different uses? Is the «school rationality» different from «everyday rationality»? What are the consequences in the educational sphere? Further research can be devoted to examining these important points more deeply.

REFERENCES

Andronico, M. (2000). Giochi linguistici e forme di vita. In Marconi, D. (Ed.), *Guida a Wittgenstein*. Laterza, Roma-Bari, 241-288.

Apel, K.-O. (1987). Fallibilismus, Konsenstheorie der Wahrheit und Letzbegründung. Forum f. Philosophie. *Philosophie und begründung*. Suhrkamp, Frankfurt a.M., 116-211.

Bagni, G.T. (1997). *Elementi di storia della logica formale*. Pitagora, Bologna.

Bocheński, J.M. (1956). *Formale Logik*. Verlag Karl Alber, Freiburg-München (page numbers refer to the Italian translation: *La logica formale.* Einaudi, Torino 1972).

Bonomi, A. (1973). *La struttura logica del linguaggio*. Bompiani, Milano.

Brandom, R. (1994). *Making it Explicit*. Harvard University Press, Cambridge MA.

Brandom, R. (2000). *Articulating Reasons. An Introduction to Inferentialism*. Harvard University Press, Cambridge MA.

Cassirer, E. (1958). *Philosophie der symbolischen Formen*. WBG, Darmstadt.

Crombie, A.C. (1995). Commitments and styles of European scientific thinking. *History of Sciences* 33, 225-238.

D'Amore, B. (2001a). Une contribution au débat sur les concepts et les objets mathématiques. *Scientia Paedagogica Experimentalis* XXXVIII, 1, 17-46.

D'Amore, B. (2001b). Conceptualisation, registres de représentations sémiotiques et noétique: interactions constructivistes dans l'apprentissage des concepts mathématiques et hypothèse sur quelques facteurs inhibant la dévolution. *Scientia Paedagogica Experimentalis* XXXVIII, 2, 143-168.

D'Amore, B. (2003a). La complexité de la noétique en mathématiques ou les raisons de la dévolution manqué. *For the learning of mathematics* 23, 1, 47-51.

D'Amore, B. (2003b). The noetic in mathematics. *Scientia Pedagogica Experimentalis* XXXIX, 1, 75-82.

Davidson, D. (1990). The Structure and Content of Truth. *The Journal of Philosophy* 87, 279-328.

Dummett, M. (1993). Language and Communication. In *The Seas of Language*. Oxford University Press, Oxford, 166-187.

Duval, R. (1998). Signe et objet (I). Trois grandes étapes dans la problématique des rapports entre représentation et objet. *Annales de Didactique et de Sciences Cognitives* 6, 139-163.

Frege, G. (1892). Über Sinn und Bedeutung. *Zeitschrift für Philosophie und philosophische Kritik* 100, 25-50.

Frege, G. (1992). Logische Untersuchungen, Dritter Teil: Gedankegefuge. *Beitrage zur Philopsophie des Deutschen Idealismus* 3, 36-51 (page numbers refer to Italian translation: *Ricerche logiche*. Guerini, Milano 1992).

Furinghetti, F. & Radford, L. (2002). Historical conceptual developments and the teaching of mathematics: from philogenesis and ontogenesis theory to classroom practice. In English L.

(Ed.), *Handbook of International Research in mathematics education*. Erlbaum, Hillsdale, 631-654.

Gadamer, H.G. (2005). *Linguaggio*. Laterza, Roma-Bari.

Geymonat, L. (1970). *Storia del pensiero filosofico e scientifico*. Garzanti, Milano.

Habermas, J. (1983). *Moralbewusstsein und kommunikatives Handeln*. Suhrkamp, Frankfurt a.M. (page numbers refer to Italian translation: *Etica del discorso*. Laterza, Roma-Bari 1985).

Habermas, J. (1999). *Wahrheit und Rechtfertigung. Philosophische Aufsätze*. Suhrkamp, Frankfurt a.M. (page numbers refer to Italian translation: *Verità e giustificazione*. Laterza, Roma-Bari 2001).

Ilyenkov, E. (1977). The concept of the ideal. In *Philosophy in the USSR. Problems of Dialectical Materialism*. Progress Publishers, Moskwa, 71-79.

Kambartel, F. (1996). Universalität, richtig verstanden. *Deutsche Zeitschrift für Philosophie* 44, 249.

Kneale, W.C. (1962). *Modality "De Dicto" and "De Re"*. In Nagel, E., Suppes, P. & Tarsky, A. (Eds.), *Logic, Methodology and Philosophy of Science*, I. The University Press, Stanford, 622-633.

Kneale, W.C. & Kneale, M. (1962). *The Development of Logic*. Clarendon Press, Oxford.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press, Chicago (page numbers refer to Italian translation: *Metafora e vita quotidiana*. Bompiani, Milano 1998).

Lolli, G. (2005). *QED Fenomenologia della dimostrazione*. Bollati Boringhieri, Torino.

Lo Piparo, F. (2003). *Aristotele e il linguaggio*. Laterza, Roma-Bari.

Marconi, D. (2000a). Il Tractatus. In *Guida a Wittgenstein*. Laterza, Roma-Bari, 15-58.

Marconi, D. (2000b). Transizione. In *Guida a Wittgenstein*. Laterza, Roma-Bari, 59-102.

Meinong, A. (1904). Über Gegenstandstheorie. In Meinong, A., Ameseder, R. & Mally, E., *Untersuchungen zur Gegenstandstheorie und Psychologie*. Barth, Leipzig, 1-50 (page numbers refer to Italian translation: *Teoria dell'oggetto*. Quodlibet, Macerata 2003).

Messeri, M. (2000). Seguire la regola. In Marconi, D. (Ed.), *Guida a Wittgenstein*. Laterza, Roma-Bari, 151-192.

Origgi, G. (2000). *Introduzione a Quine*. Laterza, Roma-Bari.

Orilia, F. (2002). *Ulisse, il quadrato rotondo e l'attuale re di Francia*. ETS, Pisa.

Penco, C. (2004). *Introduzione alla filosofia del linguaggio*. Laterza, Roma-Bari.

Prior, A.N. (1955). *Formal Logic*. Oxford University Press, London.

Putnam, H. (1992). Significato, riferimento e stereotipi. In Bottani A. & Penco, C. (Eds.), *Significato e teorie del linguaggio*. Franco Angeli, Milano.

Quine, W.V.O. (1960). *Word and Object*. MIT Press, Cambridge MA.

Radford, L. (1997). On psychology, historical epistemology and the teaching of mathematics: towards a socio-cultural history of mathematics. *For the Learning of mathematics* 17(1), 26-33.

Radford, L. (2003a). On the epistemological limits of language. Mathematical knowledge and social practice in the Renaissance. *Educational Studies in Mathematics* 52(2), 123-150.

Radford, L. (2003b). On Culture and Mind. A post-Vygotskian Semiotic Perspective, with an Example from Greek Mathematical Thought, In Anderson, M. & Al. (Eds.), *Educational Perspectives on Mathematics as Semiosis*. Legas, Ottawa, 49-79.

Rorty, R. (1979). *Philosophy and the Mirror of Nature*. Princeton University Press, Princeton NJ (page numbers refer to the German translation: *Der Spiegel der Natur*. Suhrkamp, Frankfurt a.M. 1981).

Rorty, R. (1994). Sind Aussagen universelle Geltungsansprüche? *Deutsche Zeitschrift für Philosophie* 42, 6, 975-988.

Ryve, A. (2004). Can collaborative concept mapping create mathematical productive discourses? *Educational Studies in Mathematics* 26, 157-177.

Russell, B. (1905). On Denoting. *Mind*, 14, 479-493.

Russell, B. (1910). Knowledge by Acquaintance and Knowledge by Description. *Proceedings of the Aristotelian Society* 11, 108-128.

Schnädelbach, H. (1992). Thesen über Geltung und Wahrheit. *Zur Rehabilitierung des animal rationale*. Suhrkamp, Frankfurt a.M., 104-115.

Sfard, A. & Kieran, C. (2001). Cognition as communication. Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, Activity*, 8, 1, 42-76.

Sonesson, G. (1998). The Concept of Text in Cultural Semiotics. *Trudy po znakyvym sistemam - Sign System Studies 26*, Tartu University Press, Taru, 88-114.

Strawson, P.F. (1950). On Referring. *Mind*, 59, 320-344 (Flew, A., Ed.: *Essays in Conceptual Analysis*. Macmillan, London 1960, 21-52).

Vattimo, G. (Ed.) (1993). *Enciclopedia Garzanti di Filosofia*. Garzanti, Milano.

Von Wright, G.E. (1951). *An Essay in Modal Logic*. North-Holland, Amsterdam.

Wartofsky, M. (1979). Perception, representation and the forms of action: towards an historical epistemology. In *Models. Representation and the scientific understanding*. Reidel, Dordrecht, 188-209.

Williams, M. (1996). *Unnatural doubts*. Princeton University Press, Princeton NJ.

Wittgenstein, L. (1922). *Tractatus logico-philosophicus*. Routledge and Kegan Paul, London.

Wittgenstein, L. (1953). *Philosophische Untersuchungen*. Blackwell, Oxford.

Wittgenstein, L. (1956). *Bemerkungen über die Grundlagen der Mathematik*. Blackwell, Oxford.

Wittgenstein, L. (1969a). *Philosophische Grammatik*. Blackwell, Oxford (page numbers refer to the Italian translation: *Grammatica filosofica*. La Nuova Italia, Firenze 1990).

Wittgenstein, L. (1969b). *Über Gewissheit*. Blackwell, Oxford.