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Sets, Symbols and Pictures: A Reflection on Euler Diagrams in Leonhard Euler's Tercentenary (2007)

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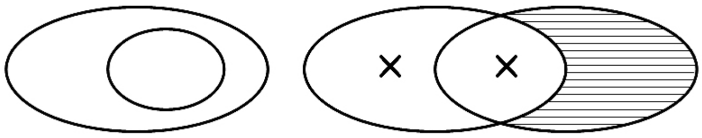
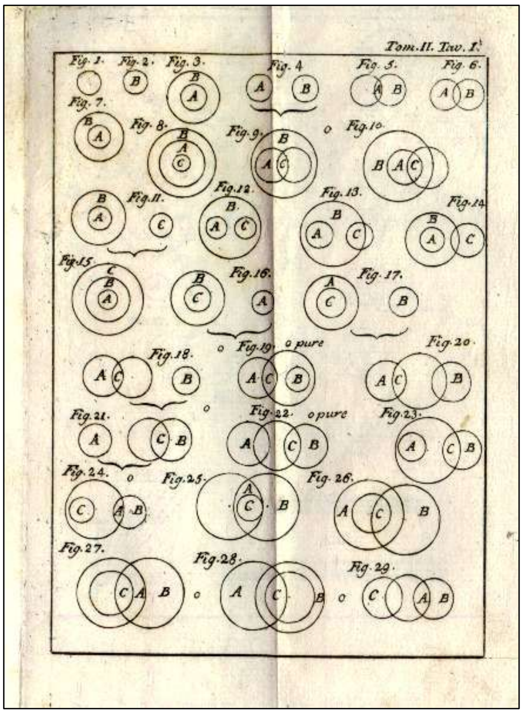
ABSTACT***:*** *The basic expressions of Set Theory are rooted in the linguistic structure of subject-predicate, and Euler-Venn diagrams should not be seen as means to replace the meaning of the predicative structure. In this paper we propose and discuss an example in order to show that the traditional representation by Euler-Venn diagrams cannot be considered, from the educational viewpoint too, completely equivalent to verbal or symbolic predicative expressions.*

In general, different systems of representation embody the students' spatial and temporal mathematical experience in a variety of ways (Radford, 2002 and 2003a; Bagni, 2006) and the use of a particular representation links different conceptual aspects (an experimental research about the role of representations in students' understanding of sets is discussed in: Gagatsis & Al., 2001). With reference to representative registers, Duval's use of the term "register" emphasizes the particular operational character of signs (Duval, 1995; it is worth noting that this idea of register is different from Halliday's: for the latter, a register is defined as a linguistic variety based on use linking the text, the linguistic and the social systems: Halliday, 1985; in this work the term "register" will be used in Duval's sense). We are going to propose a brief reflection on some features of the educational representation by Euler-Venn diagrams.

In this paper we shall not propose a complete outline of the history of sets representations (see for instance: Baron, 1969). Nevertheless, let us notice that sometimes the denomination *Euler diagram* is referred to a diagram that does not visualize all the possible intersections of the considered sets, while a *Venn diagram* is a diagram in which all the possible intersections are visualized. For instance, in the following picture, a set is visualized as a (proper) subset of another set according to Leonhard Euler (1707-1783), on the left, and to John Venn (1834-1923), on the right (a part with "×" is certainly non-empty; a shaded part is certainly empty):



Euler's representation was published in 1772 (see the Fig. 1, from the book *Lettere ad una Principessa d'Alemagna*, Ferres, Napoli 1787, the first Italian edition of *Lettres à une princesse d'Allemagne*; in the Fig. 2 we show the title-page of the work).

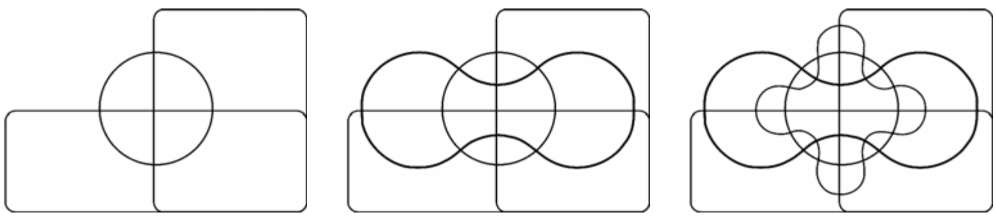


LETTERE
AD UNA
PRINCIPESSA
D'ALEMAGNA
SOPRA DIVERSI SOGGETTI
DI
FISICA E DI FILOSOFIA
SCRITTE DA M.r EULERO,
E TRADOTTE DAL FRANZESE
CON AGGIUNTA DI NOTE
DALL' ABATE
ORONZO CARNEVALE
TOMO PRIMO.

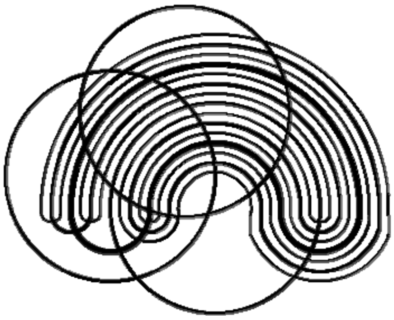
IN NAPOLI
Presso i Fratelli Terres.
Con Licenza de' Superiori.
MDCCLXXXVII.

Fig. 1 **Fig. 2**

From the educational viewpoint, it is well-known that Euler diagrams are particularly clear and intuitive. Concerning Venn diagrams, let us show in the following picture Edwards' constructions referred to 3, 4, 5 sets (Edwards, 2004):



Original construction by Venn (1880) is the following:



Let us now consider the following example:

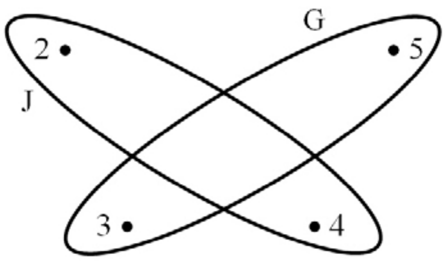
A = {1, 5}; B = {1; 2}; C = {2; 3}; D = {3; 4}; E = {4; 5};

F = {2; 5}; G = {3; 5}; H = {1; 4}; I = {1; 3}; J = {2; 4}

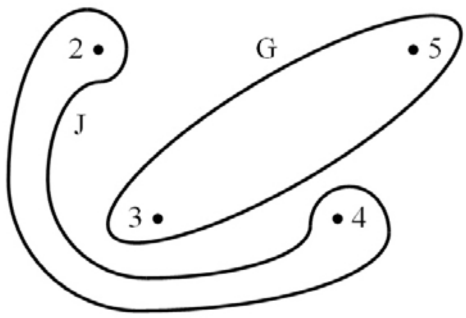
We are going to represent these sets by an Euler-Venn diagram. Let us point out two basic remarks:

(1) If we write A = {1, 5}, B = {1; 2}, C = {2; 3}, D = {3; 4}, E = {4; 5}, F = {2; 5}, G = {3; 5}, H = {1; 4}, I = {1; 3}, J = {2; 4} we have to "connect" every element of {1; 2; 3; 4; 5} with each of the other elements of the same set. In other words, in order to trace an Euler-Venn diagram of a set wit two elements, we must realize a sort of graphic "link" of the considered elements (in general, a set is represented by a connected plane figure):

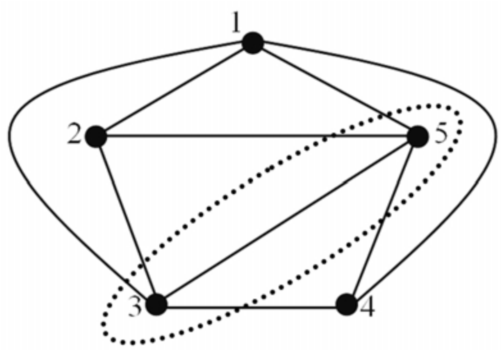
(2) In order to avoid misunderstandings and mistakes, of course it is preferable that two disjoint sets are represented by disjoint plane figures (in an Euler diagram). For instance, the following representation of the sets G and J would cause misunderstandings to a pupil (as a matter of fact, is their intersection empty?):



Of course, in order to represent clearly the sets G = {3; 5} e J = {2; 4}, we can change the position of the points-elements; nevertheless, if we keep the original positions, the (unusual, but quite correct) representation can be the following:



Let us now turn back to the exercise previously proposed. We have to trace a *complete and planar* (see previous remarks) *graph with 5 points*; but it is well-known that the complete graph K_5 (one of Kuratowski's graph) is *not* planar. For instance, with reference to the following figure, let us trace the nine sets to which the nine couples of elements linked by a segment belong (e.g. the set G = {3; 5} is represented by the dotted line).



Now it is impossible to trace a connected representation of J = {2; 4} without graphical "intersections" that, from the educational viewpoint, can cause some misunderstandings (for instance, the wrong statement of a non-empty intersection of G = {3; 5} and J = {2; 4}).

So the situation described by A = {1, 5}, B = {1; 2}, C = {2; 3}, D = {3; 4}, E = {4; 5}, F = {2; 5}, G = {3; 5}, H = {1; 4}, I = {1; 3}, J = {2; 4} cannot be represented by Euler diagrams without unusual graphic features. Of course we can ask ourselves: is connection a compulsory characteristic for Euler-Venn diagrams? Even if one does not consider the connection absolutely necessary, it is clear that a graphic representation of a set by two disconnected parts would induce several students to refer to two sets (see moreover: Chilakamarri, Hamburger & Pippert, 1996).

Can we conclude that traditional Euler-Venn diagrams (and by this we mean diagrams traced by using connected figures) have an epistemological status that is completely different from the status of symbolic or verbal expressions, referred to the relation of belonging? Surely these systems of representation (in particular, Euler diagrams) have different features, and express different information. It is worth noting that every kind of notation and representation implies some ties: for instance, when I draw a circle, during the resolution of a geometric problem, I introduce a tie: I should not be able to draw another circle whose diameter is 1/1000000 of the diameter of the circle previously drawn, although of course this is a situation that, from the theoretical viewpoint, can be considered. In cases like this it is necessary to overcome difficulties by some cunning; so it is important to realize the presence of those ties, from the educational viewpoint, too.

However, in our opinion, we can state that different representation registers are not "equivalent"; and the students were "affected" by signs (Radford, 2002) in the sense that signs offered them new paths of conceptual development (Gagatsis & Al., 2001; Bagni, 2006). Since the functioning of our mind (following: Duval, 1995) is indivisibly linked to the existence of several representation registers, it is very important to develop students' ability to use and to co-ordinate different registers (Bagni, 2005); nevertheless we must reflect carefully about this co-ordination: is it always possible? Completely or not?

And we can finally ask ourselves: *what* do these (different) representations represent? Propositions, according to Richard Rorty, cannot be conceived (only) as simple expressions of our experience, or as representations of an extra-experiential reality, a reality "out there", with our mind as a great mirror containing various representations (Rorty, 1979); they are sequences of signs and sounds used by human beings in order to develop social practices (Rorty, 1992), and representations oh mathematical "objects" can be framed into this approach. Every kind of representation has typical features, contains particular kinds of information, and is connected to particular uses, to particular social practices (Radford, 2003b). It is wrong, and useless, to make reference to different systems of representation as different forms of a universal "mathematical language" by which the "mathematical objects" can be introduced into our mind.

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