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Didactics and history of numerical series: Grandi, Leibniz and Riccati, 100 years after Ernesto Cesaro's death

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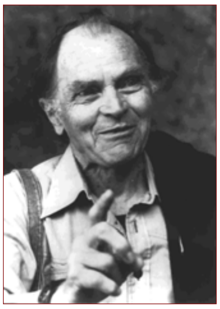
Summary

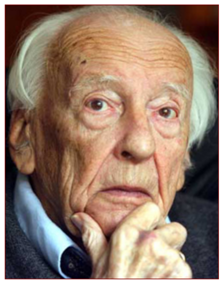
- **The approach to history**: three quotations
- **Series in the 18th century**: Grandi, Leibniz, Riccati
- **History and didactics**: an educational experience
- **An *ironic* mathematics**: Frobenius and Cesàro
- **Concluding remarks**: "calculating thought"

Three quotations: Feyerabend



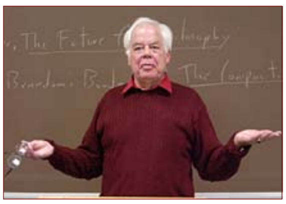
- It is very important «to frame our ideas and our conceptions of the world in an historical perspective», in **Paul Karl Feyerabend**'s words.
- He underlines moreover: «this task is not simple, because **our vision of the history is influenced by some models that hypnotize us**» (from *Lezioni Trentine*, Lectures in Trento, p. 17).

Three quotations: Gadamer



- According to **Hans-Georg Gadamer**, «*to think historically* actually means *to carry out completely the transposition that concepts of the past go through* when we try to think on the basis of them. [This] always implies a **mediation** between [historical] concepts and our thinking» (*Truth and Method*, pp. 809-811).

Three quotations: Rorty



- **Richard Rorty** notices that *irony* brings us to think that nothing has an intrinsic nature, an essence. As a consequence we are induced to believe that the presence of terms 'just', 'scientific', 'rational' in our current vocabulary is **not** a good reason to state that «the research of the essence of justice, science and rationality […] will bring us **beyond our current language games**» (*Contingency, Irony, and Solidarity*, p. 91).

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Guido Grandi (1671-1742)

- In 1703, Guido Grandi (1671-1742) noticed that from 1–1+1–1+1–1+... it is possible to obtain "both" the "results" 0 and 1:
- (1–1)+(1–1)+(1–1)+... = = 0+0+0+… = 0

  1+(–1+1)+(–1+1)+... = = 1+0+0+… = 1
- The "sum" of the series 1–1+1–1+1–1+... was considered ½ by Guido Grandi.

Guido Grandi (1671-1742)

- A (wrong!) "proof" would be:

  1–1+1–1+... = *s*

  1–(1–1+1–1+…) = *s*

  1–*s* = *s* so finally: ***s* = ½**
- According to Grandi, a relevant argument would be:

  $$\frac{1}{1+x}=\sum_{i=0}^{+\infty}(-x)^i=1-x+x^2-x^3+...$$

  (of course, nowadays we know that it requires: $-1<x<1$)

In fact mentioned "proofs" of 1–1+1–1+… = 1/2 *do not* work!

- However…
- … we have to take into account the following important issue: **did the term "convergence" (with its modern meaning) belong to Grandi's vocabulary?**
- So could we propose a correct historical analysis of Guido Grandi's series on the basis of the notion of convergence?
- And what about educational implications?
- Let's now consider another historical reference…

Leibniz studied Grandi's series

- **Gottfried Wilhelm Leibniz** (1646-1716) studied Guido Grandi's series in some letters (1713-1716) to the German philosopher **Christian Wolff** (1678-1754), where Leibniz introduced the **"probabilistic argument"** (that influenced, for instance, Johann and Daniel Bernoulli).
- If we "stop" the infinite series 1–1+1–1+... (Leibniz, 1716, p. 187), it is possible to obtain both 0 and 1, with the same **"probability"**.

Leibniz studied Grandi's series


EXCERPTA
EX ACTIS ERUDITORUM
TOMI QUINTI SUPPLEMENTORUM.
EPISTOLA G. G. L.

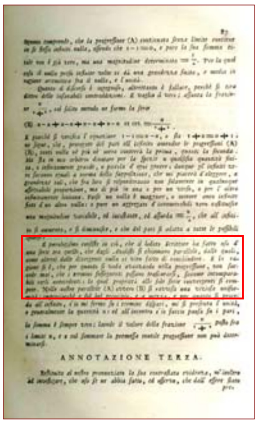
- As a matter of fact, «the *series finita* […] can have an **even** number of terms, and the final one is negative: 1–1, or 1–1+1–1 […] or an **odd** number of terms, and the final one is positive: 1, or 1–1+1». Leibnitian conclusion is the following: «when numbers' nature vanishes, our possibility to consider even numbers or odd numbers vanishes, too. [So] we ought to take the arithmetic mean [of 0 and 1], i.e. the half of their sum; and in this case **nature itself respects *justitiae* law**» (Leibniz, p. 1716, 187).

Riccati criticized Grandi (and Leibniz)

- Forty years later, **Jacopo Riccati** (1676-1754) criticised the convergence of Grandi's series to ½; in his *Saggio intorno al sistema dell'universo* (1754), he wrote: «[Grandi's] argument is interesting, but wrong. […] The mistake is caused by the use of a series […] from which it is impossible to get any conclusion, [because] it does not happen that the following terms can be neglected in comparison with preceding; **this property is verified only for convergent series**» (vol. I, p. 87).

Riccati criticized Grandi (and Leibniz)

- In fact, Jacopo Riccati made reference to some fundamental keywords referred to **convergence**.
- We can say that Jacopo Riccati's vocabulary is clearly different from Grandi's one.


ANNOTAZIONE TERZA.

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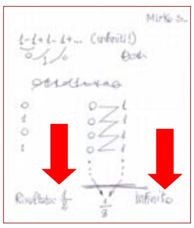
An educational experience (students aged 16-18)

- Let us now briefly consider our students' opinions regarding Guido Grandi's series.
- A test (Bagni, 2005) has been proposed to students of two third-year *Liceo Scientifico* classes, total 45 students (aged 16-17 years), and of two fourth-year *Liceo scientifico* classes, 43 students (aged 17-18 years; total: 88 students), in Treviso (Italy).
- Their mathematical curricula were traditional: in all classes, at the moment of the test, students **did not know** infinite series.

An educational experience (students aged 16-18)

- We asked our students to consider the expression "1–1+1–1+..." (studied "in 1703" by "the mathematician Guido Grandi"), taking into account that "addends, infinitely many, are always +1 and –1" and to express their own "opinion about it" (time: 10 minutes; no books or calculators allowed).
- Some students stated that the "sum" of the considered series is ½ and they made reference to justifications similar to Leibnitian "probabilistic argument".
- Let us now briefly consider, for instance, Mirko's protocol.

An educational experience (students aged 16-18)



- Visual elements (the **line** dividing "finite" and "infinite") is meaningful: at the beginning we have a sequence of 0 and 1.
- Final situation (**"infinite"**) is different: we have not "two" numbers: we have to write **a single value** after the arrows: the arithmetic mean, ½.
- The role of **didactical contract** is important: it forced the student to write a single **"result"**.
- Audio-recorded material allowed us to point out a salient short passage (1 minute and 35 seconds, 9 utterances):

An educational experience (students aged 16-18)

[1] Researcher: "Why did you write that the result is ½?"
[2] Mirko: "Oh, well, I start with 1, so I have 0, then 1, 0 and so on. There are infinitely many +1 and –1."
[3] Researcher: "That's true, but how can you say ½?"
[4] Mirko: "If I add the numbers, I obtain 1, 0, 1, 0 and always 1 and 0. The mean is ½."
[5] Researcher: "And so?"
[6] Mirko: "The numbers that I find are 1, 0, and 1, 0, and 1, 0 and so on: clearly, for every couple of numbers, one of them is 0 and one of them is 1. So these possibilities are equivalent and their mean is ½."
[7] Mirko: [*after 12 seconds*] "Perhaps my answer is strange, or wrong, but I don't see a different correct result: surely both the results 0 and 1 are wrong. If I say that the result is one of that numbers, for instance 1, I forget all the other numbers, an infinite sequence of 0."
[8] Researcher: "So in your opinion both 0 and 1 cannot be considered the correct answer."
[9] Mirko: "Alright, and in this case what is the result? I wrote that ½ is the result of the operation because ½ is the mean, so it is a number that, in a certain sense, contains both 0 and 1."

An educational experience (students aged 16-18)

- Let us consider two interesting expressions:
  [4] "If I *add* the numbers…"
  [9] "And in this case what is *the result*?"
- So Mirko makes reference to algebraic procedures: a series, in his opinion, is a kind of **algebraic operation.** It is necessary **"to add"** the numbers in order to obtain the (one and only) **"result"**.
- Let us remember that Grandi's series has been expressed in the form "1–1+1–1+...", with the remark "addends, infinitely many, are always +1 and –1": **the language is algebraic** so several students made reference to "algebraic rules".

An educational experience (students aged 16-18)

- Mirko did not make explicit reference to probability: **he just tried to find out a result for the considered problem, and this is an educational issue (clearly influenced by the didactical contract);** in the 18th century, the probabilistic argument was based upon a different remark.
- What is, nowadays, the correct reaction to be assumed by the teacher?
- To state "Grandi's series converges" is **wrong**; but…
- …our reaction, as we shall see, would require **"irony"** (in the sense of Richard Rorty's *Contingency, Irony, and Solidarity*, pp. 89-90).

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The "sum" of nonconvergent series

- Of course Grandi's series is indeterminate.
- Nevertheless it… "converges", for instance, in the sense of **Georg Frobenius** (1849-1917).
-  This notion is based upon ideas of **Daniel Bernoulli** (1700-1782) and **Joseph Raabe** (1801-1859), and has been generalized by **Ludwig Otto Hölder** (1859-1937) and **Ernesto Cesàro** (1859-1906).

Grandi's series and convergence according to Frobenius (Cesàro)

- With reference to the series $a_0+a_1+a_2+\ldots$, let us consider the sequence of partial sums:
  $s_0 = a_0 \quad s_1 = a_0+a_1 \quad s_2 = a_0+a_1+a_2 \quad \ldots$
- Being $n\geq 0$ let us say σ_n the **arithmetic means** of $s_0, s_1, \ldots, s_n$.
- The considered series converges according to Frobenius if $\boldsymbol{\sigma_0, \sigma_1, \sigma_2, \ldots}$ converges.
- If we now consider Grandi's series, $\sigma_0, \sigma_1, \sigma_2, \ldots$ is:
  1, 1/2, 2/3, 1/2, 3/5, 1/2, 4/7…
  and **it converges to ½.**

So… take care!

- The statement "Grandi's series does not converge" requires **"irony"**: i.e., in Richard Rorty's words, a subject's frame of mind to discuss his/her own vocabulary, and **the awareness that this vocabulary is not «closer to reality than others»** (Rorty, *Contingency, Irony, and Solidarity*, pp. 89-90).
- Let us think to our series: partial sums calculated, for instance, after 3, 5, 7 terms: 1–1+1, 1–1+1–1+1, 1–1+1–1+1–1+1 etc. **are (always) 1.**
- Nevertheless, according to Frobenius and Cesàro it would be possible "to distinguish" these different situations: **arithmetic means are 2/3, 3/5, 4/7…**

So…
take care!

- Of course we do not suggest just to "compare" two notions of convergence
- (let us underline that in the 20th century some important techniques based upon mentioned ideas of Frobenius and Cesàro have been applied to Fourier series).
- However the considered example can suggest a wide educational approach, **according to which different experiences that give sense to mathematical language are correctly considered.**

So…
take care!

- It is well-known that algebraic procedures are very important for mathematical learning. But some students link to (and perhaps identify in) the algebraic language some **processes**: so, for instance, if a series is expressed by algebraic signs… has it to be considered as an algebraic procedure?
- This behaviour requires a critical revision: **algebraic procedures have to be correctly considered (used).**
- Further research can be devoted to the study of the influence of algebraic language:
  – with reference to the legitimation of procedures;
  – and to different kinds of argumentation.

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The main problem of the passage from finite to infinite is a cultural one…

- … and **historical issues are important in order to approach it, although, undoubtedly, the historical approach is to be considered together with other educational approaches** (see: Radford, 1997).
- This problem requires an appropriate **mediation** (Gadamer, 2000, 811),
- and, as we noticed, it can take into account Rortian **"irony".**

An ironic mathematics
and Heidegger's "calculating thought"

- In fact, an "ironic mathematics" can be "mathematically" very deep.
- And it can induce our students to "open their eyes", to have a look, awarely, at mathematics itself, and at the world, without the problems connected to technicality; without, in **Martin Heidegger**'s words, the influence of **«calculating thought».**

(a very rare smile of Heidegger…)

THANK YOU FOR YOUR KIND ATTENTION

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